Archimedes on Conoids

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While it seems generally accepted that Menaechmus was the discoverer of the curves now referred to as conic sections, there is less agreement about which geometer should be credited with the formulation of the elementary theory of the sections. Both Archimedes and Apollonius made liberal use of Euclid's Elements and extended his now lost book on Conics. Apollonius approach seems to have been the more methodical and rigorous style of an academician. Archimedes was a critic of those who could not prove their assertions but he had a "whatever it takes" pragmatic style as exemplified by this comment from the introduction to "The Method"

"…for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge."

For the purpose of showing the evolution of the theory it seems appropriate to start with Archimedes who used the old names and definitions found in Book XI of Euclid's Elements where Euclid describes the construction of a cone of his day.

> Def. 18. When a right triangle with one side of those about the right angle remains fixed is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a cone. And, if the straight line which remains fixed equals the remaining side about the right angle which is carried round, the cone will be right-angled; if less, obtuse-angled; and if greater, acute-angled.
>
> Def. 19. The axis of the cone is the straight line which remains fixed and about which the triangle is turned.
>
> Def. 20. And the base is the circle described by the straight line which is carried round.

Conic sections were created from such a cone by cutting it with a plane that was perpendicular to an edge of the cone. The different sections, created by varying the vertex angle of the cone, were identified by the cone angle - "section of a right-angled cone," "section of an obtuse-angled cone," or, "section of an acute-angled cone," etc.

In On Conoids and Spheriods, Archimedes begins with this definition written in the terminology of Euclid's definition given above.

DEFINITION.

> If a cone be cut by a plane meeting all the sides [generators] of the cone, the section will be either a circle or a section of an acute angled cone [an ellipse]. If then the section be a circle, it is clear that the segment cut off from the cone towards the same parts as the vertex of the cone will be a cone. But, if the section be a section of an acute angled cone [an ellipse], let the figure cut off from the cone towards the same parts as the vertex of the cone be called a segment of a cone. Let the base of the segment be defined as the plane comprehended by the section of the acute angled cone, its vertex as the point which is also the vertex of the cone, and its axis as the straight line joining the vertex of the cone to the centre of the section of the acute angled cone.

Archimedes' use of "section of an acute angled cone" here is simply the phraseology of his day to describe what we now know as an ellipse as Heath indicates by inserting [ellipse] in his translation of the text. We make that substitution here to simplify our discussion.

Rephrased DEFINITION.

> If a cone be cut by a plane meeting all the sides of the cone, the section will be either a circle or an ellipse. If then the section be a circle, it is clear that the segment cut off from the cone towards the same parts as the vertex of the cone will be a cone. But, if the section be an ellipse, let the figure cut off from the cone towards the same parts as the vertex of the cone be called a segment of a cone. Let the base of the segment be defined as the plane comprehended by the elliptical section, its vertex as the point which is also the vertex of the cone, and its axis as the straight line joining the vertex of the cone to the centre of the elliptical section.

The gist of this definition then is that when a plane cuts through all sides of a cone the section produced will be either a circle or an ellipse. If it is a circle, the part of the cone between the plane and the apex is still a cone. But, when the cut is an ellipse then the part of the cone between the plane and the apex becomes a segment of a cone with the section as its base.

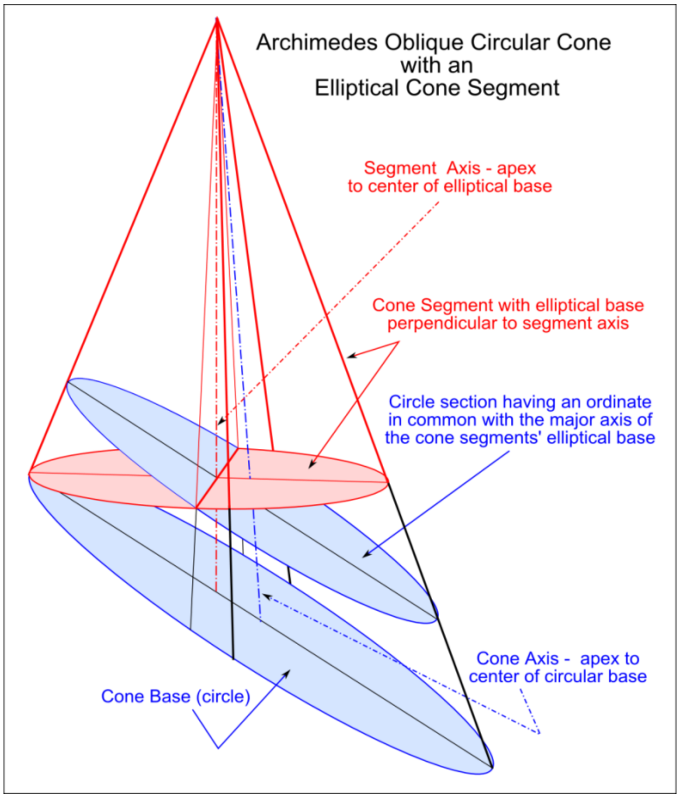
In today's terminology Archimedes' "cone" is a "circular cone" and his "segment" an "elliptical cone." Context determines whether "cone" is being used in the sense of Archimedes' "segment" or his "cone". "Right" and "oblique" distinguish between cones with the apex centered over the base and those where it isn't. Hopefully while discussing Archimedes' work I will succeed in consistently using his terminology.

The axis of either a cone or a segment is the line from the apex to the center of the base. A segment axis will not, however, be collinear with the axis of the cone from which it was cut.
The lines (generators) from the apex to the opposite ends of a diameter of the base form an axial triangle. Because the axes are not collinear the axial triangles of a segment are not the same as those of its parent cone.

When the axis is perpendicular to the base the planes of the axial triangles will be perpendicular to the base. Those in a cone will also be isosceles and similar while those of a segment will only be similar in pairs as an ellipse has a rotational symmetry of two. The axis lies along the intersection of the planes of the triangles.

The situation becomes more complex when the axis is not perpendicular to the base. In a cone, the triangle formed from the shortest and longest generator lines will be the only one lying on a plane that is perpendicular to the base plane. The minor axis of the cone's cross section (the section perpendicular to the apex bisector) lies on this plane while the major axis lies on a plane perpendicular to it - the plane of the isosceles triangle. Thus, the "cone" axis lies both on the "perpendicular triangle" plane, which is perpendicular to the base, and on the "isosceles plane" which is not. Hence, it can be said that the cone axis lies on a plane that is perpendicular to but is itself not itself perpendicular to the base.

Cone segment bases, however, are not circular and thus the segment axis need not lie on a plane that is perpendicular to the base of the segment. The orientation of the segment axis can be used to categorize segments into three groups. Those with the segment axis perpendicular to the segment base; those with the axis lying on a plane that is perpendicular to the base but is itself not perpendicular to the base; all other orientations of the axis with respect to the segment base. Archimedes only considers the first two categories. The first in Proposition 7 and the second in Proposition 8.



The figure on the following page is as close as we get to dealing with an actual cone. It, however, is really just a perspective drawing – an illustration – of the entity described in Archimedes' definition. While it helps to conceptualize the definition it offers little quantitative information. The Greek Geometers worked with even simpler two dimensional models composed of circles, triangles and straight lines. For these to make sense and to be of use requires some understanding of the both the geometric shapes themselves and the entities they represent.

Consider the blue circle in the figure that intersects with the elliptical segment base. In a 2D illustration the circle looks no different than the elliptical base. And, though it is known that the major axis of the elliptical base is perpendicular to the paper, how can any quantitative information be determined?

The center of the elliptical sections lie on the symmetry axis, the bisector of the apex angle, while the centers of circle sections lie on either the cone axis, or, its reflection in the symmetry axis, the symmedian.

Assume that the cone was constructed first and that the elliptical section that became the base of the segment was then cut so that it is known to be perpendicular to the segment axis. The axis also passes through the center of the segment base and bisects the apex angle. Hence the segment base is also the cross section of the cone. The major axis of the cone cross section is parallel to the base of the cone making it perpendicular to the paper in agreement with the notation in the figure.

Now for the blue circle. Any section of the cone that is parallel to the cone base is also a circle. When a circle intersects an elliptical section the intersection is along a common chord. When this chord is the major axis of the ellipse, it will not be a diameter of the circle because their centers do not lie on corresponding ordinates. The center of circle sections parallel to the cone base all lie on the cone axis. Those that lie on the symmedian are known as the anti-parallels.

The ratio of the ordinates of an ellipse to the corresponding ordinate of the ellipse's circumcircle is a constant $\frac{CB}{CA}$. Using the theorem that "If two straight lines drawn in fixed directions between two lines forming an angle intersect in a point, the ratio of the rectangles under the segments is independent of the positions of the point" means that any ordinate can be used and the ratio calculated from the product of the components of the minor axis divided by the product of the components of the circle diameter.

Finding the cone when the section is given is more difficult but possible. We look first at Archimedes solution when the segment axis is also known.

Proposition 7.

Given an ellipse with centre C, and a line CO drawn perpendicular to its plane, it is possible to find a circular cone with vertex O and such that the given ellipse is a section of it {or, in other words, to find the circular sections of the cone with vertex O passing through the circumference of the ellipse].

Conceive an ellipse with BB' as its minor axis and lying in a plane perpendicular to that of the paper. Let CO be drawn perpendicular to the plane of the ellipse, and let O be the vertex of the required cone. Produce OB, OC, OB', and in the same plane with them draw BED meeting OC, OB' produced in E, D respectively and in such a direction that

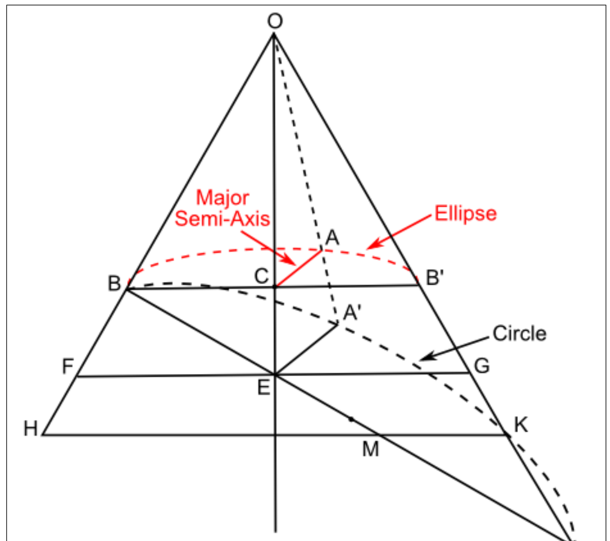
$$\frac{BE.ED}{EO^2} = \frac{CA^2}{CO^2}$$

(where CA is half the major axis of the ellipse.) And this is possible since

$$\frac{BE.ED}{EO^2} > \frac{BC.CB'}{CO^2}$$

Now conceive a circle with BD as diameter lying in a plane at right angles to that of the paper, and describe a cone with this circle for its base and vertex O.

We have therefore to prove that the given ellipse is a section of the cone, or, if P be any point on the ellipse, that P lies on the surface of the cone.



Archimedes On Conoids Proposition 7

Heath notes that both the construction and the proof are assumed known. We thus will follow the proposition statement as far as it goes and then discuss three possible methods for completing the construction.

Archimedes begins constructing the required cone by first constructing a segment with the given ellipse as its base with the given line OC perpendicular to it. Connecting OB and OB' completes the segment. This "segment" is then cut by a plane perpendicular to the paper along the line BD drawn through point E so as to produce a circular section and thereby construct an circular cone containing the given ellipse as a section.

If the generator line OA is extended to OA' then the triangles OCA and OEA' are similar and $\frac{EA'}{OE} = \frac{CA}{OC}$. But A' is to be a point on and *EA'* an ordinate of a circle constructed on BD. When this is accomplishedEA'^2will = $BE * ED$. After squaring both sides of the first equation, substituting $BE * ED$ for EA'^2 and rearranging, we end up with

$$\frac{CA^2}{OC^2} = \frac{EA'^2}{OE^2} = \frac{BE * ED}{OE^2}$$

This is the equality that Archimedes says must be satisfied by the line BD drawn through point E so that the point A' will lie on the circle constructed on BD. In the proof we have omitted, Archimedes demonstrates that any other point on the circle lies on the cone.

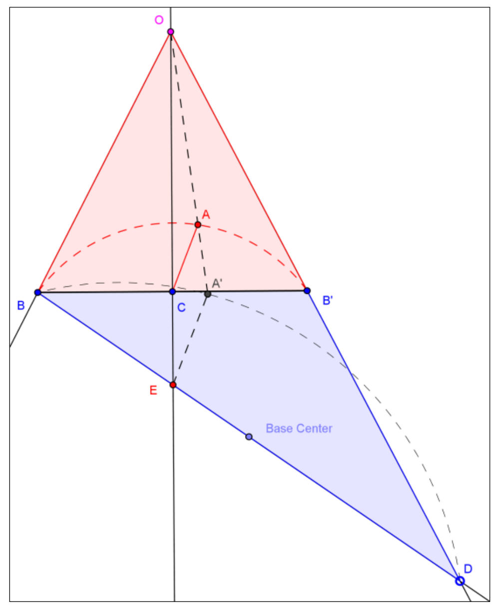
The problem thus amounts to drawing a line BD through a point E; measuring OE, BE and ED; calculating the quotient $\frac{\mathrm{BE{*}ED}}{OE^2}$; compare quotient with $\frac{CA^2}{OC^2}$. If the quotients are equal BD is the required line. If not then chose a new point E and repeat the steps until the required line is found.

This would be a rather tedious process to carry out manually and it can be simplified by choosing point C as the pivot. This eliminates one variable and appears to result in a simplified manual process.

Since Archimedes assumed the construction was known it may well be that he knew a pure geometrical construct to find the required cone. However, I cannot find evidence that a pure geometrical approach was known.

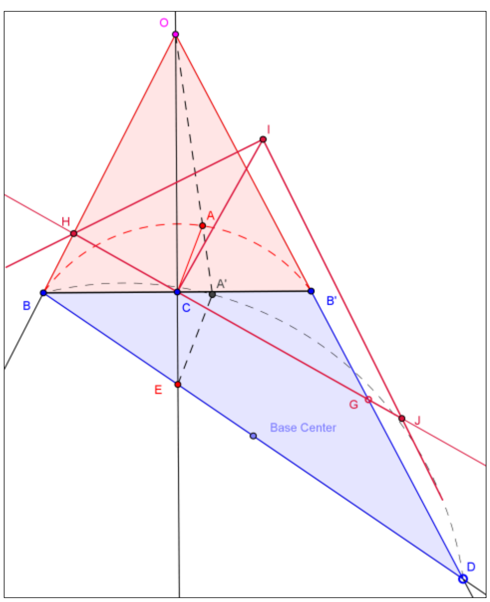
When finding a cone containing a given section, Apollonius finds only cones having the apex over the center of the base. Sections having their major axis parallel to the base are not found in such cones. Consequently Apollonius rotates the section so that the minor axis is parallel to the base and then finds a cone containing the rotated section.

However Archimedes did it, it is easy to demonstrate the two mechanical solutions and a "stationary" geometric solution with GeoGebra and a GeoGebra model that does so accompanies this document.



The first model allows the required cone to be found interactively by dragging point D along OD while E is constrained to move on OC extended. Sliders are provided for setting values of CB, CA and OC. Then as D is drug along the extended leg of the segment the two quotients in the required equality are calculated. . Point E changes from black to red when the equality is satisfied. When this occurs, OBD will be the required cone and BD will be its circular base.

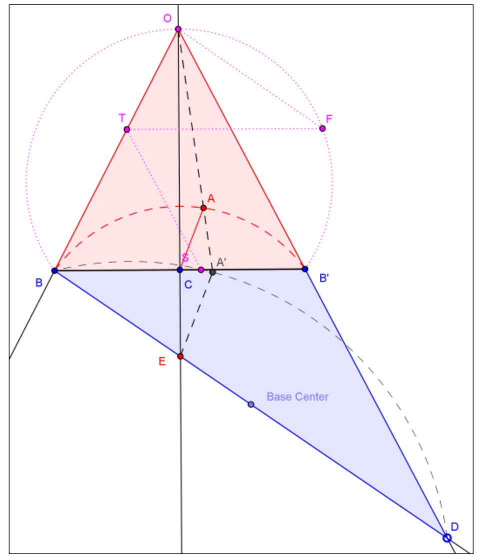
Once you see how things work, drag point D so that it lies between point B' and the apex and observe that at some point, the equality will again be true. You have located the sub-contrary circles (anti-parallels to the cone base circle). They are at the same angle with respect to the elliptical base as the circular base but on the opposite side. The elliptical segment base is still the cross section (perpendicular to the apex bisector) of the cone OBD and it bisects the angle between the plane of the circular base BD and that of the sub-contrary section just found.



The GG model contains a second method using a right angle and a straight edge to find a circle section. The line HG is drawn through and pivots about point C while H and G are constrained to move along OB and OB' respectively. CI is constructed, with a length of CA, at point C and perpendicular to HG. HI is drawn and then IJ is drawn perpendicular to it through point I with J constrained to move on HG extended. In this arrangement the right angle HIJ pivots about point I when point G is drug to move HGJ but remains a right angle even as HC, HG, HI and IJ change in length and point J slides along HG extended.

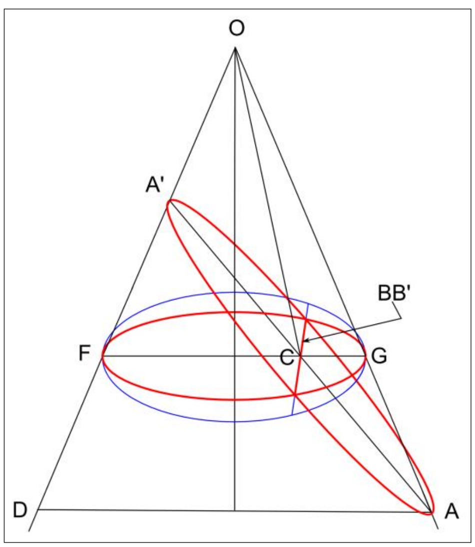
Pivoting HCG about C makes OE = OC which reduces the required equality to $CA^2 = HC * CG$. But $CA = CI$ and $CI^2 = HC * CJ$. Thus to find a circle section we need only to move J by dragging G until J and G become common. The section for constructing the required cone can now be found by drawing the line BED parallel to HJ.

The third model is a "stationary" geometric model requiring only that the CA, CB, and OC sliders be set.



First find the point S such that $BS = \frac{CB^2}{CA^2} BB'$. Then draw OBB', the circumcircle of triangle OBB'. Draw a line parallel to OB' through point S and intersecting OB at T. Draw TF parallel to BB' through T and intersect the circumcircle at F. Draw OF and then BED parallel to OF. BED will be the required circular base of cone OBD.

If you have read in Apollonius' Conics you may have observed that this has some similarities to what he does in Book I proposition 56 (I.56). However, Apollonius only finds right circular cones in I.56 and sections with their major axis parallel to the base (landscape aspect ratio) do not exist in right circular cones. Apollonius handles such sections by rotating them so that the major axis is perpendicular to the base (I.58) and finding that section instead.



In proposition 8 Archimedes finds a cone having O as its apex that contains a given elliptical section and a given line OC not perpendicular to it.

The given ellipse is to be supposed to lie in a plane perpendicular to the plane of the paper with AA' its major axis and BB' the other axis. The lines OA and OA' are drawn and extended to create a segment as in Proposition 7.

As created, the segment axis is not perpendicular to the segment base and the point of interest here is this step from Archimedes construction.

"Conceive a plane through AD perpendicular to the plane of the paper, and in it describe either (a), if $CB^2 = FC*CG$, a circle with diameter AD, or (b), if not, an ellipse on AD as axis such that, if d be the other axis, $\frac{dd'^2}{AD^2} = \frac{CB^2}{FC*CG}$*."*

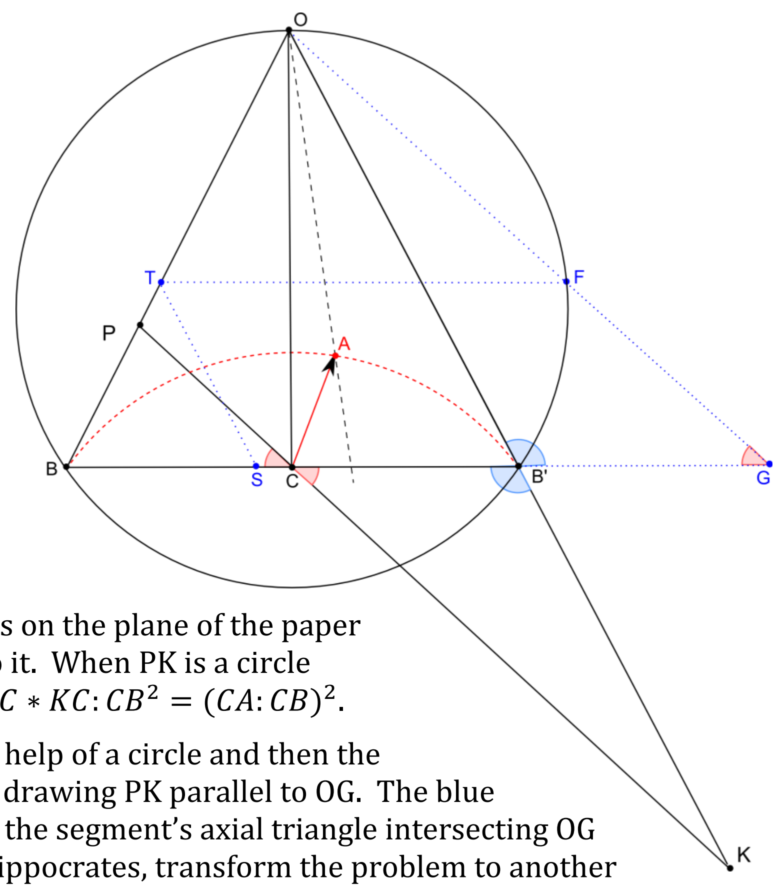
Archimedes cuts a section FG through C, the midpoint of the ellipse, and perpendicular to the apex bisector and tests it to see if it is a circle. Because he requires that AA' be the major axis of the elliptical section CB will be equal FC*CG if FG is the diameter of a circle and AD cut parallel to FG will be circular making ODA the required cone.

If CB^2 <> FC*CG what Archimedes proposes is a clear indication that he understood the geometry of cones and their sections. A circle having as a diameter the axis of an ellipse is referred to as an auxiliary circle and the ratio of the lengths of the corresponding ordinates of the ellipse and its auxiliary circle is a constant. CB is here a common ordinate of the sections AA' and FG and is collinear with an ordinate of the auxiliary circle FG. Hence the ratio of the ordinates of the section FG to the corresponding ordinates of its auxiliary circle FG can be found and is $\frac{CB^2}{FC*CG}$. Archimedes scales this on the section diameter AD as $\frac{dd'^2}{AD^2} = \frac{CB^2}{FC*CG}$ and makes it the base of the segment ODA.

Heath assumes the construction method from Proposition 7 can now be used to find the cone. Even so, this seems to be the end of the line for "moving" geometric solutions at least until interactive graphics became available. Rather that pursue it farther, in the next section a transition is made to the "stationary" geometric methods that Apollonius later use. Whether Apollonius knew of Archimedes work or developed his theories from Euclid's Elements independently is not known.

We begin the transition by revisiting Proposition 7. Recall that the segment OBB' was elliptical and the segment axis OC was perpendicular to it. The minor axis lies in the plane of the paper and the major axis is perpendicular to it. The purpose of Prop. 7 was to find a circular section in the segment that can be used to make a cone from the given cone segment.

We propose here to show how such a circular segment can be found without having to use moving geometry. Starting with the same cone segment as before, we draw a line from the segment apex, O, extending it to intersect BB' extended at G. PK is then a section created by cutting the segment with a plane that is perpendicular to the paper and parallel to the line OG.



Triangle PCB is similar to triangle OGB which makes $PC{:}\,OG = CB{:}\,BG$. Triangle CB'K is similar to triangle GB'O making $KC{:}\,OG = CB'{:}\,B'G$.

Multiplying the two equalities gives:

$PC * KC{:}\,OG^2 = CB * CB'{:}\,BG * B'G$ which when rearranged becomes

$PC * KC{:}\,CB^2 \;=\; OG^2{:}\,BG * B'G.$

From the proposition it is known that the minor axis of the segment base, CB, lies on the plane of the paper and the major axis, CA, is perpendicular to it. When PK is a circle $PC * KC = CA^2$. Thus, $OG^2{:}\,BG * B'G = PC * KC{:}\,CB^2 = (CA{:}\,CB)^2$.

Fortunately point G can be found with the help of a circle and then the geometrical solution can be completed by drawing PK parallel to OG. The blue circle passes through the three vertices of the segment's axial triangle intersecting OG at F. Now rather than finding G, we like Hippocrates, transform the problem to another but, fortunately, one for which there is a solution.

From Euclid it is known that the products of the components of secants that intersect externally are equal and thus, $BG * B'G = OG * FG$. Substituting this into the equation above and inverting both sides above yields $(CA{:}\,CB)^2 = OG^2{:}\,BG * B'G = OG^2{:}\,OG * FG = OG{:}\,FG$ which reduces to $FG{:}\,OG \;=\; (CB{:}\,CA)^2$.

Finding F is easily done using parallel lines since lines intersected by parallel lines are divided in the same proportions. From B find a point S on BB' so that $BS{:}\,BB' = (CB{:}\,CA)^2$. Draw a line through S parallel to OB' and intersecting OB at T. This divides OB so that $TB{:}\,OB = BS{:}\,BB'$. Draw a line through T parallel to BB' intersecting the circle at F. Draw a line from O through F intersecting BB' extended at G. This divides OG so that $FG{:}\,OG \;=\; TB{:}\,OB \;=\; BS{:}\,BB\,' = (CB{:}\,CA)^2$

Thus, any section cut parallel to OFG produces a circular section. This is the method underlying the geometric solution given in Proposition7.

There is an interactive GeoGebra model of the three solutions on this website.

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