Catapults and Cube Roots

Fearing a long war with Carthage, Dionysius, the ruler of Syracuse, funded a large effort aimed at developing novel weapons in the Early 4th century BC. Out of this effort came the first mechanization of archery - machines for shooting arrows using wooden and composite bows that were more powerful than handheld ones.

When the limits of flexible bows was reached even more powerful devices using torsional springs were designed. Their first reported use in the mid 4th century set off an arms race that led to ever increasing "monster" devices. Philon mentions both Rhodes and Alexandria as development centers. In particular he credits Ptolemy's support for their being a systematic method to catapult design.

Ctesibius is considered the founder of the Alexandrian school of mathematics and engineering and Philo(n) of Byzantium, an engineer and a writer , is thought to have succeed him. Philo authored the *Mechanicus*, a nine book treatise of which only four books have survived. Much of what is known about catapults comes from Philo's 4th book, *Belopoeica* (on catapults (artillery)). Among other things, Philo gives credit to the incentives and subsidies that Ptolemy gave the Alexandrian technicians for coming up with the design criteria for calibrating the new breed of "rock throwers."

> *... through the analysis of former mistakes and the observation of subsequent experiments, the fundamental principle of the construction was reduced to a constant element, the diameter of the circle holding the spring. This was first done by Alexandrian technicians who benefited from large subsidies from fame-seeking kings who supported craftsmanship and technique.…*

The catapults in consideration used torsional springs and for the "rock throwers" the optimal diameter of the hole through which the spring was threaded, and hence the diameter of the spring, was proportional to the cube root of the weight of the projectile. A lot of trial and error experimentation was needed before Alexandrian technicians determined the design formula that Philon describes thusly:

> *Reduce to units the weight of the stone for which the engine must be constructed. Make the diameter of the hole of as many finger-breaths [i.e., dactyls] as there are units in the cube root of the number obtained adding, furthermore, the tenth part of the root found. If the weight's root is not a whole number, take the nearest one; but if it is the nearest above, endeavour to diminish proportionally the added tenth, and if the nearest below, to increase by a tenth.*

The dimensions of all the other parts are defined in fixed multiples of the hole diameter

Some 200 years (1st century BC) after Philon, Vitruvius, a Roman architect and military engineer who had seen military service overseeing the building of catapults, wrote a treatise that he called "*The Ten Books on Architecture.*" One book from that collection, the *Balliste*, gives a systematic method for designing torsion catapults and provides the spring sizes for a large number of different projectile weights. Vitruvius indicates that the spring is proportional in size to the projectile weight but does not give the proportionality for the reason given in this paragraph:

> *'For the holes made in the capitals through the openings of which are stretched the strings made of twisted hair, generally women's, or of sinew, are proportionate to the amount of weight in the stone which the ballista is intended to throw, and to the principle of mass, as in catapults the principle is that of the length of the arrow. Therefore, in order that those who do not understand geometry may be prepared beforehand, so as not to be delayed by having to think the matter out at a moment of peril in war, I will set forth what I myself know by experience can be depended upon, and what I have in part gathered from the rules of my teachers, and wherever Greek weights bear a relation to the measures, I shall reduce and explain them so that they will express the same corresponding relation in our weights."*

Hero of Alexandria writing in the first century AD is a third source of optimum catapult spring sizes for given projectile weights also based upon the Alexandrian formula. As is to be expected, the lists have significant differences. David Campbell summarizes what is generally accepted to be a credible cause of the discrepancies:

> *For Vitruvius's description of Roman artillery, there are no MS diagrams. He uses as module for his catapults the hole in the peritrete, just as Philon and Heron, but there is a problem about his ballista, the stone thrower. He gives us a complete list of weights of the missile, from 2 to 360 librae, and the diameters of the holes, in pedes and digiti; but his diameters are less than those of Philon and Heron by about a quarter. Marsden suggests that this is due to an error in transmission; that Vitruvius did not use the digit, 16 to a foot, but the uncia, 12 to the foot, and that the name of the measure has been changed by some scribe.*

The first two columns in the following table were taken from the aforementioned lists. The next column, the first column of the calculations, shows the value whose cube root must be found to calculate the spring diameters using the optimized formula, as given by Philon. The calculated diameters are in the second calculations column.

Tabulated by Campbell		Spring Diameter Calculations			
Stone Weight (mina)	Spring Diameter (dacty)	$100m$	$1.1\sqrt[3]{100m}$ (dactyl)	$m/10$	$11\sqrt[3]{m/10}$ (dactyl)
10	11.0	1000	11.0	1.0	11.0
15	12.6	1500	12.6	1.5	12.6
20	13.9	2000	13.9	2.0	13.9
30	15.9	3000	15.9	3.0	15.9
50	18.8	5000	18.8	5.0	18.8
60	19.9	6000	20.0	6.0	20.0
120	25.2	12000	25.2	12.0	25.2
150	27.1	15000	27.1	15	27.1
240	31.7	24000	31.7	24.0	31.7
360	36.3	36000	36.3	36.0	36.3

Taking the cube root of 1000 or even 36,000 is not a big deal for a computer but the numbers whose cube roots must be reckoned as shown in the $100m$ column are not realistic for the cube root extractors of that day. One suggestion proffered by Philo in the *Belopoeica* was this:

> *It is possible from one number, the least of those mentioned, namely the 10 mina, to determine the remaining diameters instrumentally, in accordance with the duplication of the cube, as we have explained in the first book.*

Philo wrote about the duplication of the cube in the lost 'first book" *Isagoge* (Introduction to mathematics) of his *Mechanicus*. In the *Belopoeica* he shows how designing a spring for a machine to throw a weight twice that of another machine is equivalent to solving the cube duplication problem. i.e., the ratio of the spring diameters is as the cube root of the projectile masses.

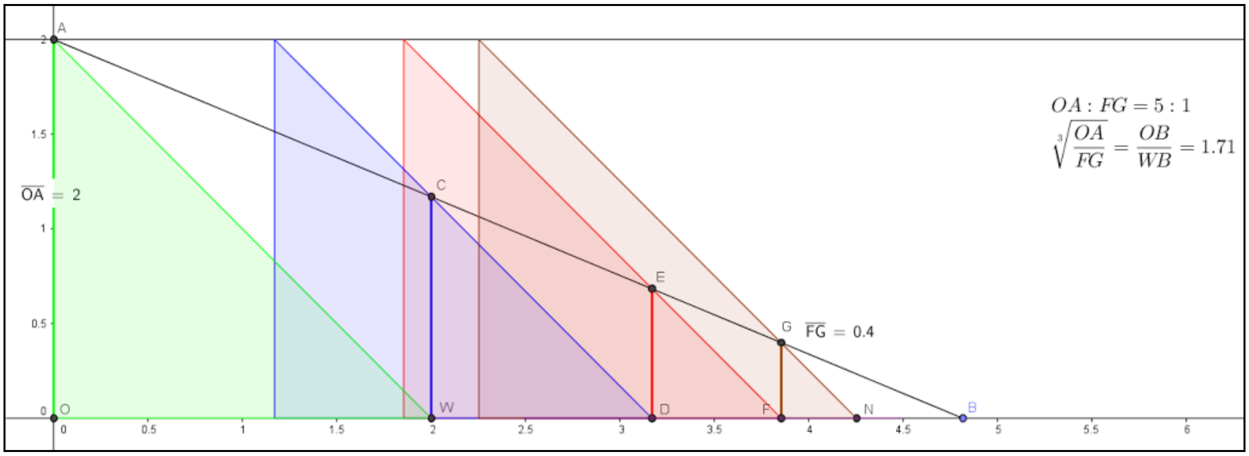
This leads to the result that $\frac{d_2}{d_1} = \frac{\sqrt[3]{2m_1}}{\sqrt[3]{m_1}} = \sqrt[3]{2}$ or, $d_2 = d_1\sqrt[3]{2}$.

Using values from the table $d_{20} = d_{10}\sqrt[3]{2} = 11 * 1.26 = 13.9$. The spring size for any weight projectile can be obtained in a similar manner using the ratio of the new projectile mass to the mass of the 10 mina device. The diameter for a 150 mina device would be: $d_{150} = 11 * \sqrt[3]{15} = \sqrt[3]{3}\sqrt[3]{5} = 11 * 1.44 * 1.71 = 27.1$.
An alternate approach is to multiply the cube root operand by $10/10$ and extract the cube root of 1000. The formula can then be refactored to $11\sqrt[3]{w/10}$ as is done in the last column of the table. The cube root operand magnitudes shown in the $m/10$ are now the same as those used with Philo's ratio method just examined. This significantly reduces the magnitude of the operand. As shown above the operand can be further reduced with factoring. For a last example,
$d_{360} = 11 * sqrt[3]36 = 11 * \sqrt[3]{3}^2\sqrt[3]{2}^2 = 11 * 1.442^2 * 1.26^2 = 36.3$ where we use cube roots from above

Philo and Eratosthenes were contemporaries but even though Eratosthenes envisioned his device as being used for catapult development Philo may not have been aware of it. Or, Philo may have agreed with Nicomedes, another contemporary, that the mesolabe was not a practical device. Nicomedes had also developed a graphical solution he called *conchoid lines*, but for whatever reason, Philo devised his own method of finding a cube root that involved a rotating ruler to draw a line that has become known as Philo's line.

Philo's solution is essentially the same as one credited to Hero of Alexandria as well as a proposed by Apollonius. They differ only in the mechanical method use to align a single line - Philo's line.

Look now at the example of Eratosthenes's mesolabe which shows it being used to find the $\sqrt[3]{5}$ as needed for the 50 mina device.



A GGB simulation of the standard "find two means between 2 and 1" device with square panels is used. The device is manipulated by dragging point B to rotate AB around point A until

$$\frac{2}{FG} = \frac{N}{1} \text{ or } FG = \frac{2}{N}.$$

For N=5 this becomes

$$FG = \frac{2}{5} = 0.4$$

and

$$\sqrt[3]{\frac{OA}{FG}} = \sqrt[3]{\frac{2}{0.4}} = \sqrt[3]{5} = \frac{OB}{WB} = 1.71.$$

To find the cube root of 10 would require setting

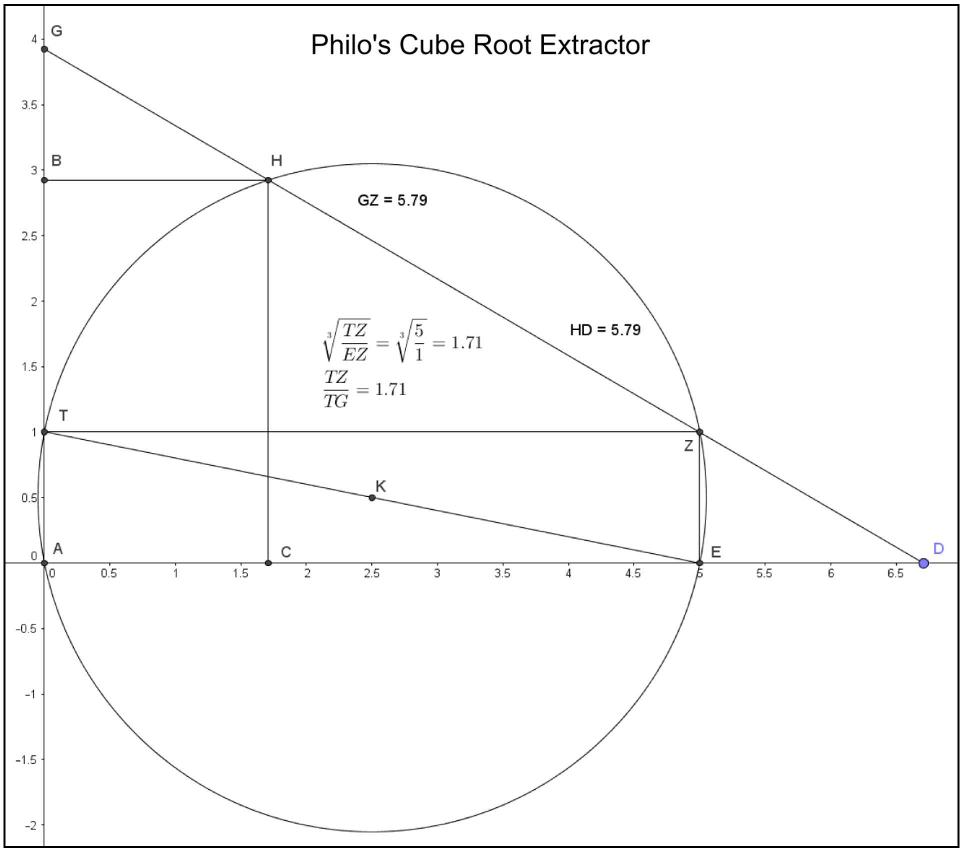
$$FG = \frac{2}{5} = 0.2$$

and for 12,

$$FG = \frac{2}{12} = 0.17.$$

A difference so small that it would seem to exceed or at least be close to the limit of what one might expect to be able to do with with a mechanical version of this device.

Philo's device was really just a compass and a ruler. In this example the device is used to extract $\sqrt[3]{5}$ as was done with the mesolabe.

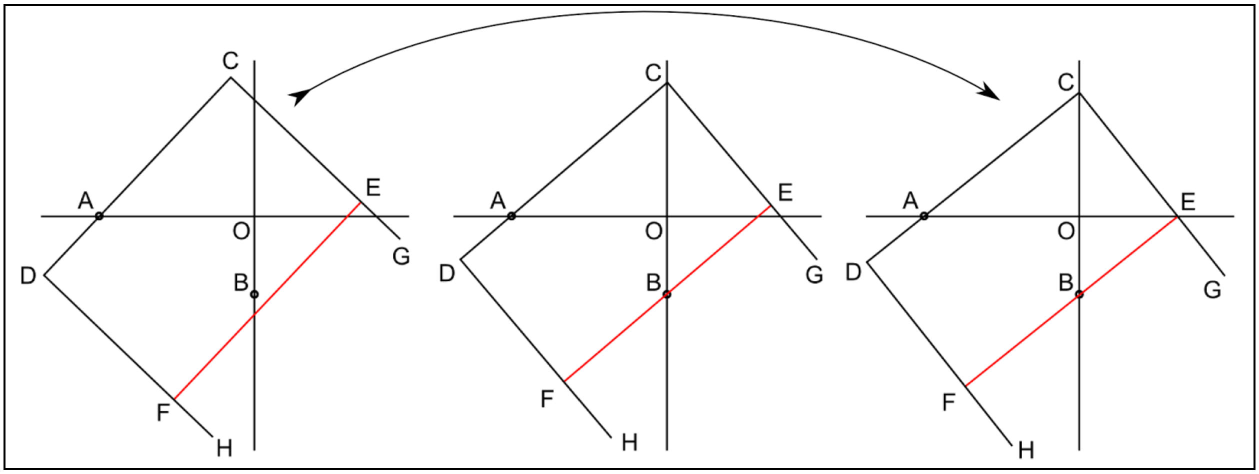


Construct a rectangle OZ with sides $\frac{TE}{EZ} = \frac{N}{1}$. Draw a circle with center at mid point K of TE and passing through Z. Place a ruler to pivot about point Z and adjust its so that GH=DZ. The coordinates of point H will then be the means between the coordinates of point Z. i.e., $TZ : CH = CH : BH = BH : ZE$

$$\sqrt[3]{\frac{5}{1}} = \sqrt[3]{\frac{TZ}{EZ}} = \frac{TZ}{CH} = \frac{CH}{BH} = \frac{BH}{ZE} = 1.71.$$

The line GD is commonly called Philo's Line. While trivially easy to position using GGB I have a distinct feeling that using the mechanical version would not be trivial or even easy. If this was the method Vitruvius used to extract a cube root, then his providing a list of precalculated spring diameters vs projectile weights is understandable. Vitruvius expressed such and opinion.

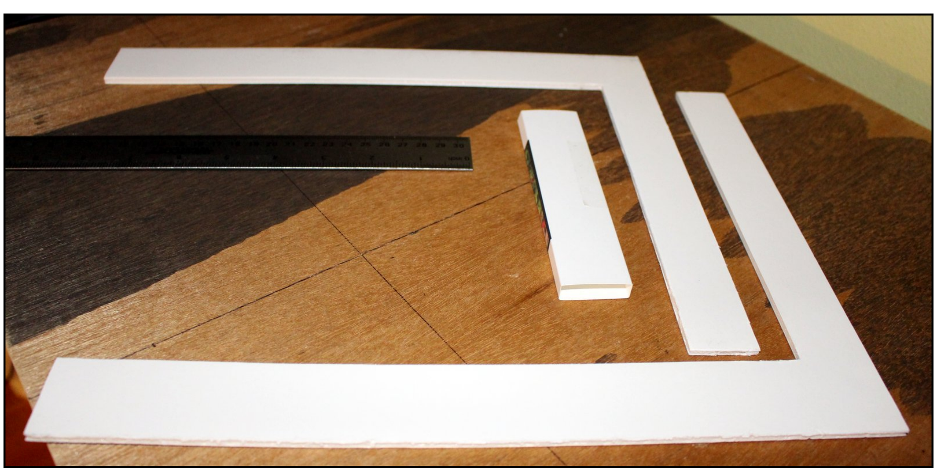
The last extractor discussed is of unknown origin but is called Plato's Frame. A simplified version whose operation is almost self explanatory could look like this.



Going from left to right, points A and B are positioned so that $\frac{OA}{OB} = \frac{N}{1}$. The frame is placed so that DC will pivot on A as it rotates clockwise. The frame is then rotated until the frame corner C lies on BO extended. The sliding bar FE is then positioned against B. With the frame held against points A and B and the corner C kept on BO extended the frame is rotated and slid downward until corner E lies on AO extended as shown in the rightmost figure.

In this configuration, $\frac{OA}{OC} = \frac{OC}{OE} = \frac{OE}{OB}$ from which $\frac{OA*OC*OE}{OC*OE*OB} = \frac{OA^3}{OB^3}$ and finally $\sqrt[3]{\frac{OA}{OB}} = \frac{OA}{OC} = \frac{OC}{OE} = \frac{OE}{OB}$.

This can easily be model in GGB for finding the cube root of 5 as was done in the first to examples. Instead, however, following Philo's lead the solution to the cube duplication solution (cube root of 2) is shown using a physical model of Plato's frame.



My device consisted of two "framing squares" cut from foam board and a rectangular tube of folded cardboard that is used to join them.

It was manipulated as shown on the preceding page until it was aligned like shown in the next figure. $OA = 2$ and $OB = 1$

$\sqrt{\frac{2}{1}} = \sqrt[3]{\frac{OA}{OB}} = \frac{OA}{OC} = \frac{OC}{OE} = \frac{OE}{OB}$. Since $OB = 1$ the simplest solution is OE.A frame with longer arms and additional holes would allow larger roots to be extracted but it would seem to still be limited. Thus none of the cube root extractors appear usable for finding roots when the operands are in the thousands.

Philon gives this example of approximating the cube root of 2000.

> *First we must find the cube root of 2,000. Now 12 cubed is 1,728 and 13 cubed is 2,197. If we choose 13, we "endeavor to diminish proportionately the added tenth". A tenth of 13 is 1.3, so that a Greek mechanic, making his rough reduction, would calculate the diameter of the hole as 13+1¼= 14¼ dactyls, or, more likely, 13+1=14 dactyls.*

This or some iterative approximation method appears to be the only feasible way to reckon hole sizes using the $d = 1.1\sqrt[3]{100m}$ formula. Using either the $d = 11\sqrt[3]{m/10}$ formula to Philo's ratio method along with factoring would also be feasible.

I suggest it likely that the initial table values were all calculated by the technicians at Alexandria. As Vitruvius indicates, there would be little need for calculations in the field and a table of cube roots would suffice for most of those and exceptional cases could be approximated. If the three roots, $\sqrt[3]{1.5}$, $\sqrt[3]{2}$ and $\sqrt[3]{5}$ are known all the holes sizes shown in the table can be calculated. Actually $\sqrt[3]{8}$ and $\sqrt[3]{1000}$ are also needed but assumed known.

It might have been appropriate for Philon and Heron to include a geometrical method for extracting roots in their manuals but it would be surprising if they used such for more than token calculations.

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