Conic Curves

Of the three geometers credited with being the first to solve the "doubling of the cube" problem, Archytas was about the same age as Plato. He was a Pythagorean leader that Plato sought out in an effort to gain an understanding of what Pythagoras believed and taught. Archytas had a mechanical bent and might well be called an applied geometer. Plato would later complain that Archytas was too concerned with searching for mathematics that could be used to explain what one could see and touch.

Archytas' solution of the cube problem was based on a standard planar model of similar right triangles for finding means but used an ingenious 3D mechanism to manipulate it.

Eudoxus was some years younger and a student of Archytas. He was acclaimed both as a mathematician and as an astronomer. Archytas is said to have established mechanics as a mathematical science and Eudoxus did the same for Astronomy. In geometry he built on the work of Theaetetus and introduced a method of proportions dealing with irrationals that had long baffled the Pythagoreans.

In Vitruvius' "Ten Books on Architecture," written about 25 B.C., Eudoxus is credited with developing the "spider's web" arrangement of lines on the face of a sundial. On a flat face dial the lines that are the trace of the sundial nodus are hyperbola but Goldstein and Bowen suggest it was more likely that Eudoxus used a bowl shaped dial where the lines would be arcs of a circle rather than hyperbola. The contention being that drawing them on a flat surface required more knowledge of conics than they believe existed at that time.

Eudoxus' is thought to have used a curve that is now called Eudoxus' Kabyle but which Eratosthenes called "the curves of the God fearing Eudoxus." Eudoxus' solution was a variant of the planar model of similar right triangles used by Archytas but incorporated the triangles into a device that was likely similar to one constructed by Descartes some two millennia later that he called a "mesolabe compass." Descartes demonstrated its use by solving the doubling of the cube problem as noted in "*Descartes's Mathematical Thought*," by C. Sasaki, footnote 71 on pages 119-120.

"For if we wish to find two mean proportionals between YA and YE, we have only to describe a circle whose diameter is YE; and because this circle cuts the curve AD at point D,YD is one of the required mean proportionals. The demonstration is obvious to the eye, merely by applying this instrument to the line YD: for, as YA, or YB which is equal to it, is to YC, so YC is to YD, and YD to YE."

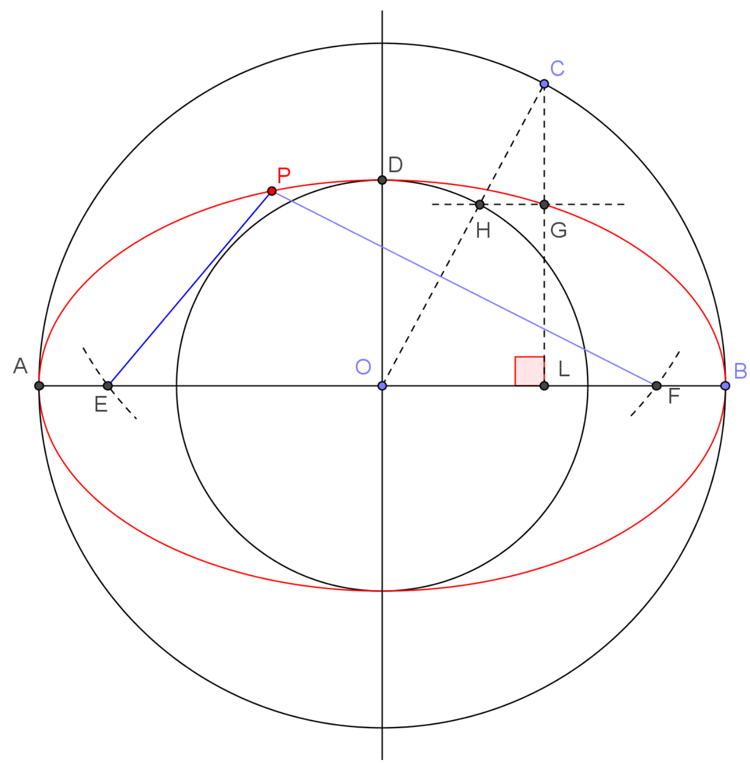
Menaechmus, younger yet than Eudoxus, was Eudoxus' student and may also have studied under Archytas. It is through Eratosthenes we know Menaechmus made use of curves that Eratosthenes described as "the triads of Menaechmus." Whether these were previously known or were discovered by Menaechmus is not known. However, it is hard to imagine that Eudoxus would not have at least wondered about the shape of the trace of the sundial nodus on a flat surface and perhaps, as a typical professor, he had suggested to his student Menaechmus that it was something he should study.

While legend has it that Archytas, Eudoxus and Menaechmus solved the doubling of the cube problem in response to the problem being presented to them by Plato, the more likely scenario would seem to be that the three solutions were conceived over an extended period of time and were reflective of the advancements they and others made in the state of the art.

Some Basics

It is not possible to know exactly what was in the journeyman geometer's toolkit in that day so this is a best guess of what might have been with a keep it simple bias. Circles, chords, right angles and parallel lines seem to be a safe and logical place to start. This section could rightly be titled, "some things I wish I had known before starting to read Apollonius' conics."

We start with a piece of string but rather than making a loop of the string by tying the two ends together in a knot, make small loops on each end so that the



string length including the loops is equal the diameter of the desired circle. Now put the two small loops over a pin; stick the pin into a cork-backed sheet of drawing paper; put a pencil in the big loop and while holding it taught draw a circle. The radius of the resultant circle is half the length of the string making the diameter the same as the string length.

Drawing the ellipse is a bit more difficult as we wish the diameter of the circle to also be the major axis of the ellipse. To accomplish this draw a diameter *AB* and and a second diameter perpendicular to it. On the second diameter mark off *OD* as the desired minor semiaxis of the ellipse. Then use *D* as the center for an arc of diameter *OA* and strike arcs on *AB* at *E* and *F* as shown in the figure.

Move the ends of the string from *O* to pins at *E* and *F*. Stretch the string with the pencil at *P* and hold it taught to draw the ellipse. Admittedly, I used GeoGebra and could draw the entire ellipse without having to contend with the string catching on the tops of the pins.

GeoGebra currently does not have an ellipse from major and minor semiaxes. It does, however, have a method for drawing an ellipse from the foci and minor semiaxis. Thus if you have the major and minor semiaxes, an arc with a radius of the major semiaxis is the only additional construction setup needed.

E and *F* are the foci of the ellipse. Foci were not something the early Greeks were familiar with when they were still grappling with doubling the cube and algebra was some two millennia in the future. Still they were able to do a rather thorough geometrical analysis and knew that the ratio of the ordinate of the ellipse to the ratio of the circle on *AB* was a constant for all ordinates.

The may have been discovered using similar triangles as we show above. A circle on the minor axis is drawn and then *OC* is drawn intersecting the incircle at *H*. A perpendicular is dropped from *C* to the diameter *AB* intersecting it at *L*. A parallel to *AB* drawn through *H* intersect *CL* and the ellipse at *G*.

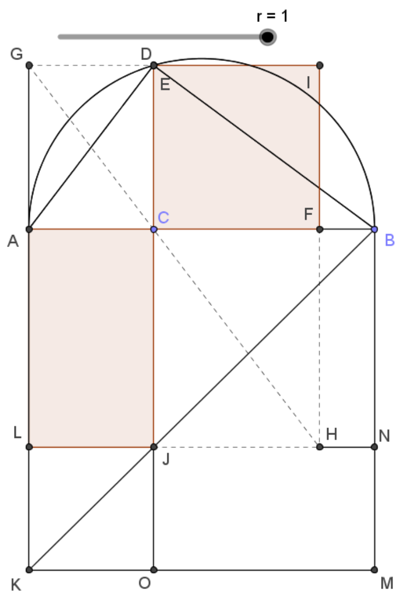
The inner circle divides the line *OC* proportional wherever *C* may be drawn and the line parallel to the diameter *AB* divides *CL* proportional so that $CG : CH :: GL : HO$. This is true for all ordinates and thus is true when *C* lies on the extension of the minor axis extended to intersect the circumcircle making the ratio of the ordinates of the ellipse to the ordinates of the circumcircle equal to the ratio of the minor axis to the major axis.

Note that the reverse of this procedure can be used to plot points to construct an ellipse and at one time was a standard method for laying them out.

Archimedes believed that it was easier to prove something if you first observed what it was you were going to prove. Essentially that is what we just did. We discovered the relationship between the ordinates of an ellipse and its major and minor axes. The next step is to offer a geometrical proof.

To that end we start with a circle which, along with right angles and triangles, had been extensively investigated by the Pythagoreans.

This figure shows a semicircle with *AB* as diameter. Triangle *ADB* is inscribed in the semicircle which makes angle *ADB* a right angle. The ordinate *CD* is perpendicular to the diameter *AB*. Thus if the circle is completed and *DC* extended it would intersect the circle *AB* at D' and *AB* would be the perpendicular bisector of the chord DCD'. Using the the product of the segments of intersecting chords in a circle theorem gives the results



$AC * CB = CD * CD' = CD^2$

showing that *CD* is the mean between *AC* and *CB*. The remainder of the figure illustrates a method of verifying the property of curves in general and will be used here to verify that the circle, of which the semicircle is half, is indeed a circle. This is done by essentially proving the product of segments of intersecting chords theorem when the chord is the double ordinate of a circle.

First *sq.EF* is constructed on *CD* with sides of length *CD*. Here points *E* and *D* are the same point but they will not always be so as will become obvious.

Next *rect.AJ* is constructed on *AC* with a side along *AK* such that the area of the rectangle is the same as the area of *sq.EF*.

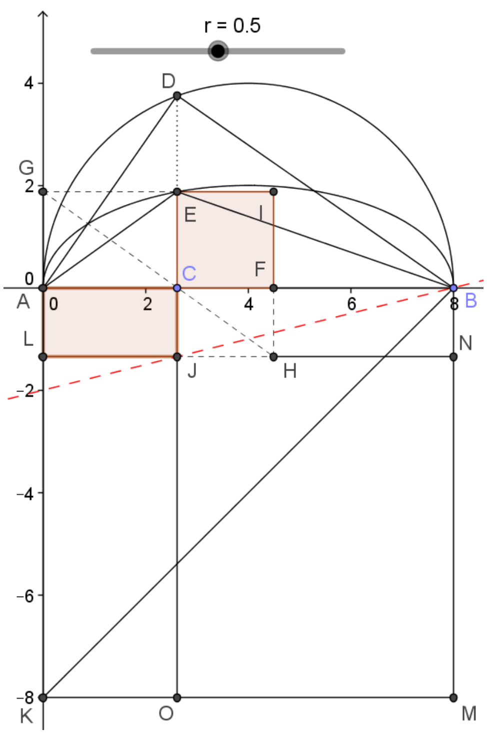
To do this, *IE* is produced to intersect *KA* extended at *G*. From *G* a line is drawn through *C* and produced to intersect *IF* extended at H. *HJ* is then drawn parallel to *CF* and extended to *L* and *N*.

Rect.AJ so constructed has the same area as *sq.EF* as can be verified by noting that *GH* is the diagonal of *rect.GH* dividing it into equal parts. *GH* also divides *rect.GC* and *rect.CH* into equal parts. Subtracting the equal parts of *rect.GC* and *rect.CH* from the corresponding equal parts of *rect.GH* leaves the area of *rect.AJ* equal the area of *sq.EF*.

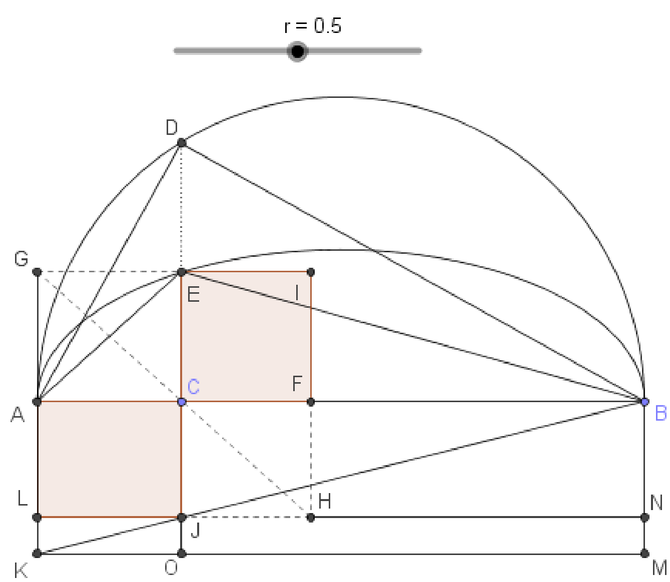
As *C* is moved along *AB*, *E* moves along the curve *ADB* and as the ordinate *CE* changes, the area of *sq.EF* and *rect.AJ* change causing *CJ* to change. *KB* is a trace of the locus of point *J* as *C* is moved along *AB* between *A* and *B* intersecting *AK* at *AK*=*AB*. *KB* is the diagonal of *sq.KB* and both *sq.KJ* and *sq.JB* lie on it. Because *CB* and *CJ* are sides of *sq.JB*, $CB = CJ$ for any point *C* on the diameter *AB*. Thus

$AC * CJ = AC * CB = CE^2 = CD^2$.

In geometric terms of that day this curve could be describe by *"The area of the square on the ordinate equals the area of the rectangle applied to the line AK with width equal that part of the diameter cut off by the ordinate and deficient by a square similarly situated as the square on CB."*
That is, when *AC=0*, *CJ=AK*. Then as *AC* increases *CJ* decreases and the ratio of the decreases in *CJ* to the increase in *AC* is *1:1* (deficient by a square) and the ratio is constant (similarly situated as the square on *CB*).



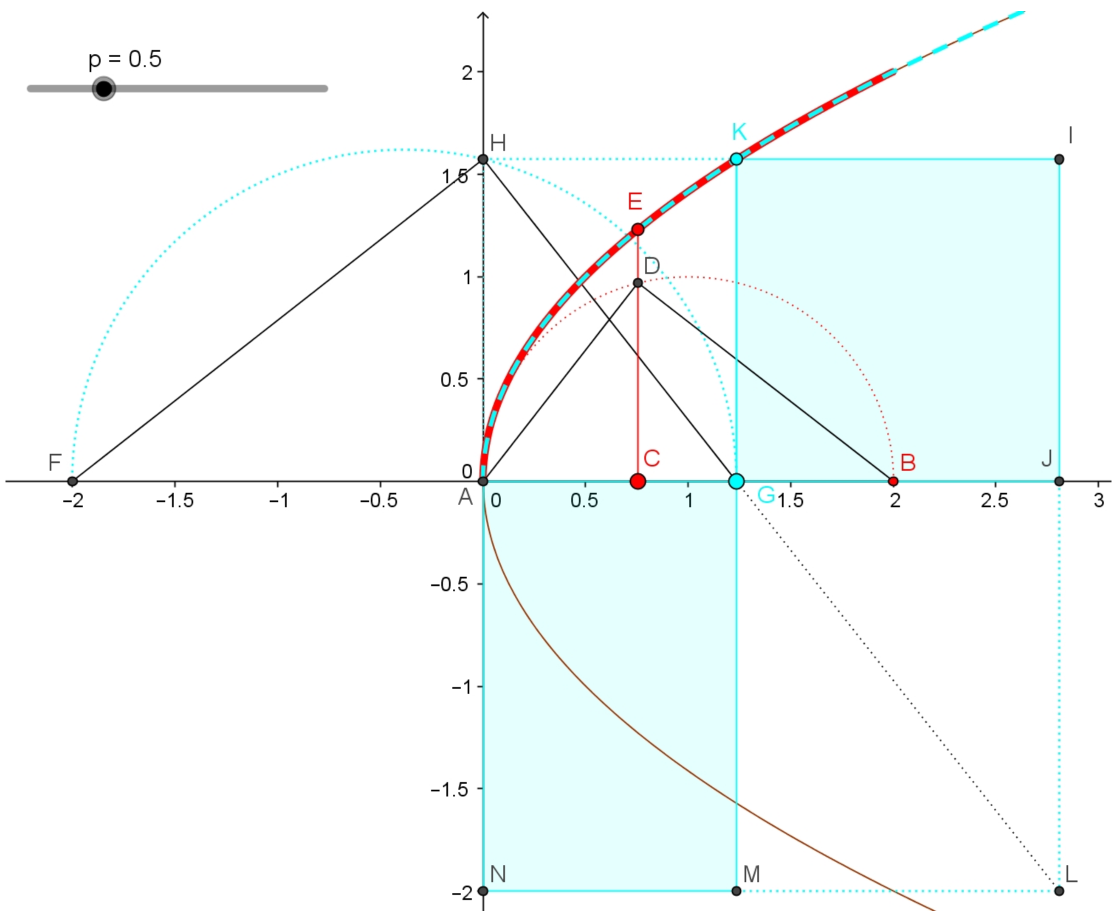
Consider next a simple example where a curve is drawn by plotting the midpoint of the circle ordinate. We might end up with something like the figure to the left where the curve *AEB* has been drawn with the ordinate of the unknown curve, *CE*, having a constant ratio, *r*, to the ordinate *CD* of the circumcircle. In this instance *r = 0.5*.
Obviously, $AK = AB$ is no longer the correct length line on which to apply the area. The dashed red line looks to be a possibility and we note that it intersects *AK* at a point *Q=2* such that $AQ : AK = AQ : AB = 0.25 = r^2$.



For a second attempt then, we set $AK = r^2 * AB$. This changes the model to that shown in the figure to the right.
Now it can be said that
"The area of the square on the ordinate equals the area of the rectangle applied to the line AK with width equal that part of the diameter cut off by the ordinate and deficient by a rectangle similar to and similarly situated to the rectangle on CB."
Notice the slight change in wording. It is now *rect.KJ* rather than *sq.KJ* and *rect.JB* rather than *sq.JB*. The change in *CJ* is no longer equal the change in *AC*

and while all squares are similar all rectangles are not. This necessitated the wording change to *"deficient by a rectangle similar to and similarly situated."*
The ratio of the change remains *AK/AB* but the ratio which was one for a circle is now a constant $r^2 = 0.25$. The ratio *r* applies to all ordinates. Thus, if *ma* is the minor axis and *MA* the major axis,
$(ma/MA) = r$ and $AK = r^2 MA = (ma/MA)^2 MA = ma^2/MA$, or, $MA * AK = ma^2$.
That is, the minor axis is the mean between *AK* and the major axis. Thus *AK* and *MA* characterize an ellipse as completely as knowing *ma* and *MA* and, when working with sections cut from a cone, *AK* is frequently easier to ascertain than is the minor axis.

The next figure is a composite of two geometric methods of generating the curve we now describe as a parabola; the same curve constructed algebraically; and an application of areas construct to classify its properties as was done with the circle and the ellipse.



The easy one of course for anyone familiar with algebra is the solid black curve which is a plot of $y^2 = 4px$. The upper branch is largely hidden by the other curves laying on it but it can be seen in the dashed cyan line.

Looking first at the red curve which appears dashed because it it overlaid with the dashed cyan curve. The right triangle *ADB* has been constructed with $AB = 4p = 2$. The dotted red semicircle is used to show that ADB is a right angle.

Using the theorem that "*the altitude on the side opposite the right angle in a right triangle divides that side proportionally to the arms of the right angle and that the arms are the mean between the side opposite the right angle and the segment adjacent to it that is cut off by the altitude and adjacent to the corresponding arm*" we see that $AD : AC = DB : CB$. But $\triangle ADC$ and $\triangle ADB$ are similar which makes $AD : AC = AB : AD$ or $AD^2 = AC * AB$. In a like manner it can be shown that *DB* is the mean between *CB* and *AB*.

CE is constructed to always equal *AD* and the dashed red line is the trace of the *l*ocus of point *E* as *C* is moved along *AB*. Notice that the curve ends when C reaches the end of the side opposite the right angle. Making the hypotenuse would result in a longer curve but it would no longer be the curve plotted by $y^2 = 4px$. (Since all parabola are similar, it would be a scaled copy of it.)

What the red curve plots is the mean between a fixed line, $AB = 4p$, and the line *AC* which varies in length from *0* to *AB* where *AB* is fixed by *p*.

The dashed cyan curve based on the right triangle *FHG* removes this limitation by using the altitude from the side opposite the right angle rather than one of the arms of the right triangle to find the mean.

In this case, $AF = 4p$ is the fixed line and *AG* is the line of variable length that can now vary from *0* to any desired length. *AH*, the altitude, is the mean between *AF* and *AG*. i.e., $AF * AG = AH^2$.

Because *AF* has a fixed length and angle *FHG* is a right angle, point *H* is forced to move upward as point *G* is moved to the right . *GK* is constructed so that it is always equal to *AH* and the cyan curve is the trace of the locus of point *K* as *G* is moved along *FA* extended.

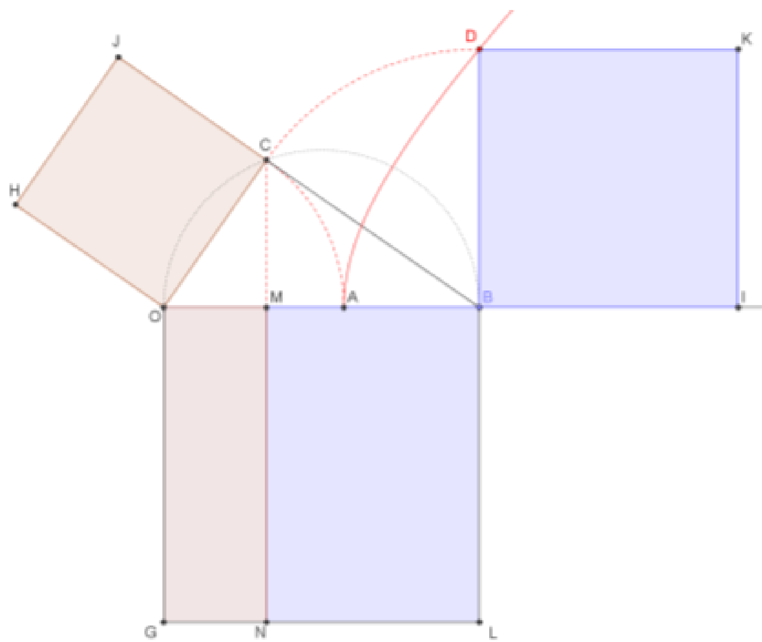
As a final step *GK* is used to find the properties of our curve. As previously done with the ellipse, we construct the square on the ordinate which in this case is *GK*. The line *KH*, parallel to *FG*, was constructed to make $GK = AH$ so we need only draw *HG* and extend it to intersect *IJ* extended at *L*. A line is then drawn parallel to *FG* intersecting *KG* extended at *M* and *HA* extended at *N*.

"The area of the square on the ordinate equals the area of the rectangle applied to the line AG with width equal the diameter cut off by the ordinate." There is no "deficient by." It is exact.

The final curve considered in this section if the one now known as a hyperbola. We look at how it can be created using a right triangle and then analyze it using the application of areas method.

The usual statement of the Pythagorean theorem is something like “the square on the side opposite the right angle is equal the sum of the squares on the sides containing the right angle.” Restated for our purposes it becomes “the square on one of the sides containing the right angle equals the difference between the square on the side opposite the right angle and the square on the other side that contains the right angle.”

In this figure, angle *OCB* is a right angle as indicated by point *OC* lying on the semicircle *OB*.



OC has some fixed value, *f*. Point *B* was initially positioned at point *A* where *OC* then lay along *OA* and *OA*=*OC*= *f* and *BC*=*0*.

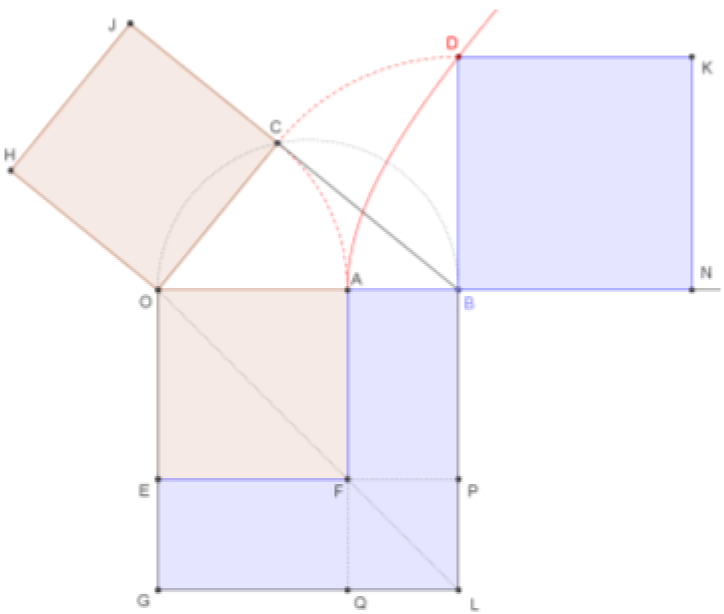
As *B* is moved to the right the length of *O*B increases as does the length of *BC*. This in turn increases the area of the squares on both *OB* and *BC*. Because angle *OCB* is fixed, the increase in *BC* causes *OC* to rotate about *O* but the square on *OC* does not change.

The square on *OB* is comprised of the square on *OC* plus that on *BC*. Thus, the square on *BC* is the difference between the square on *OB* and the square on *OC*. The square on *OB* is shown divided into areas that are equal that of the squares on *OC* and *BC*. *DB* has been constructed perpendicular to *OB* and equal *B*C. Hence the area of the square on *BD* equals the area of the square on *OB* that corresponds to the area of *BC*.

The trace of point *D* then generates a curve AD whose ordinate D at any point on the curve has a length such that the square on the ordinate equals the square on OB less the square on OC=OA=f.

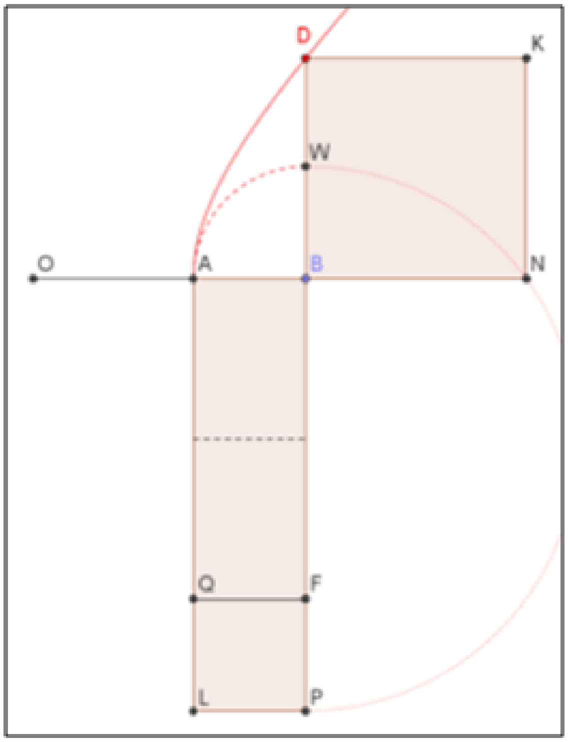
Recall that in the discussion of the parabola the theorem that in a right triangle the arms are the mean between the side opposite the right angle and the

segment adjacent to it that is cut off by the altitude. There the segment that was cut off by the altitude was fixed and the adjacent arm varied in length. Here the arm is fixed and the segment that is cut off by the altitude varies in length.
When applying the application of area method used previously to determine the characteristics of this curve, determining the line to apply the area to is not intuitively obvious. Converting the right triangle method of generating the curve to a construction method based on the application of areas simplifies the task.
To do this, restructure the area on the side opposite the right triangle. The right triangle ignores how we visualize it anyway . *Sq.OF* replaces the slice along *OG* that was the area of the square on *OC*. *AF* is extended to intersect *GL* at *Q* and *EF* to intersect *BL* at *P*. The diagonal of *OF* lies on the diagonal of the square *OL* and A*FEGLB* is a gnomon that has an area equal that of the square on the ordinate *BD* which equals the square on *BC*.



Eliminate the right triangle stuff requires constructing a rectangle on AB and AQ extended having an area equal that of the gnomon that is valid for all *AB*. You can probably visualize how this needs to be done but to make it look authentic we quote Euclid II.4
"If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts."

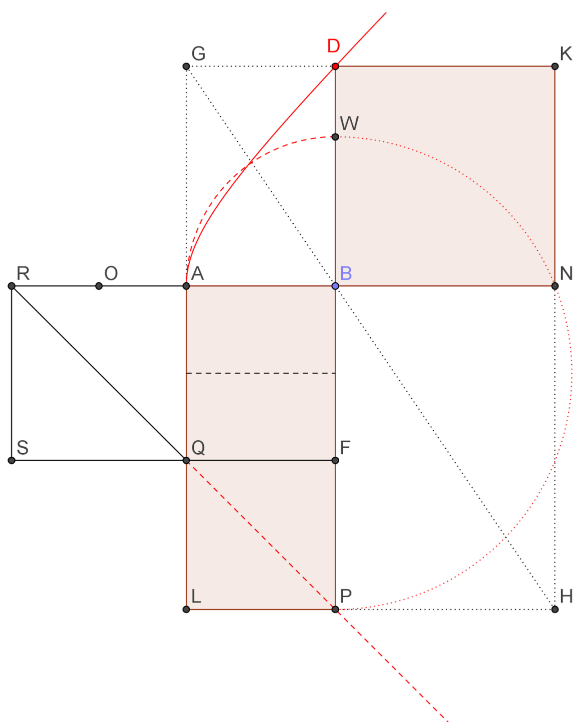
The square on *OB* can thus be viewed as consisting of the square on *OA* plus the gnomon, confirming our observation above, and the right triangle components can be replaced with two rectangles each having one side of fixed length *OA* and the other side of length *AB* plus a square with sides of length *OA*.



This reduces the model to that shown at the left where *rect.AP* consists of a *sq.Q*P with sides of length *AB* and *rect.AF* which is comprised of two rectangles with sides *OA,AB*. It is this area, *rect.AP*, that now needs to be converted into the sq.*BK* having the same area.

This can be done by drawing *arc ABW* to extend *PB* by the length *AB* to point *W*. A semicircle with *PW* as diameter is then used to find *BN* the mean between *PB* and *BW*. *BD* is then made equal to *BN*. The square on *BD* now equals the area of *rect.AP* which in turn equals the area of the square on *OB* less the square on *OA* as it did when using the sides of the right triangles to determine *BD*.

The trace of *D* is now congruent with that generated by using the right triangle procedure indicating that it was a successful replacement. The square on the ordinate is equal the rectangle on *AB* but the length of the "applied to line" must still be determined.



Experience with the two previous examples will be helpful in figuring it out. First, we trace the locus of *P* as *B* is moved and see that it lies along the diagonal of *sq.QP*. We extend it to intersect *AO* extended at *R*. and then complete the square on *RA* which is then similarly located to *sq.QP*.

We can now assert that "*the area of the square on the ordinate is equal to the area of the rectangle applied to the line AQ having as its width that part of the diameter cut off by the ordinate and exceeding it by a square similar situated as the square on RA*."

That the diagonal of *sq.QP* lies along the extended diagonal of *RQ* shows that the area of the square on the ordinate is increasing and that the area has a constant ratio to the area of the rectangle on *AB*.

In this instance we constructed a curve - a rectangular hyperbola - such that *BP* = *RB* when the area of *sq.BK* equals the area of *rect.AB,BP*. For other curves this is not necessarily the case. What is true is that

$AQ : RA = AB * BP : RB * AB = BD^2 : RB * AB$

That is the ratio AQ:RA is the ratio of the square on the ordinate to the rectangle formed by the diameter *RB* and the part *AB* that is cut off by the ordinate. For the curve used in this example this $AQ : RA = 1$ and the "exceeded by" is a square. For other ratios it will be rectangular.

Finally the points *G* and *H* and the lines *AG*, *GD*, *PH*, and *NH* needed to implement the application of area method as a check of *AQ* as the "applied to line" have been added.

If we had not just constructed the curve using rect.AP that had been constructed equal to the gnomon from the construction using a right triangle, we would not have the point *Q* that just "happened" to be where it was needed.

Instead we would have had to construct *rect.AP* from the square on *BD*. Then when the locus of point *P* was traced point *Q* would have been located. From there the process could have continued as before.

No one can say with certainty when or by whom the curves discussed in this section were first studied nor what names were given them. At some point the connection between these curves and those of sections cut from cones was recognized and likely caused some consternation as not all such sections yielded to their method of analysis.

The curves discussed thus far have a common characteristic that the ordinates were perpendicular to the diameter and, at least by the time Euclid wrote "*The Elements,*" those found "deficient by" were referred to as a section of "an acute-angled cone", those found "equal to" as section of a "right-angled cone", and those found "exceeding by" as section of "an obtuse-angled cone."

Archimedes continued to use those designations even though his definition of a cone only required that the base be round. The cone cross section could be either round or elliptical and the cutting plane need not be perpendicular to a cone edge. He did however put restrictions on the cutting plane orientation with respect to the axial triangle plane of the cone that effectively limited the sections

produced to those having perpendicular conjugate diameters which he called axis.
Apollonius would later remove that restriction and show that all sections cut from an arbitrary cone that had a circular base could be classified into one of three categories based upon the physical properties of the cone itself and the cutting plane provided that the cutting plane intersected the cone base plane in a line that was perpendicular to the axial triangle base or to its extension.
He proved this by showing that principal diameter of such sections lay along the intersection of the cutting plane with the axial triangle of the cone and that the diameter conjugate to it was parallel to the plane of the cone base.
Apollonius defined what he called the “parameter” using the properties of the cone and the cutting plane angle and used it as the “applied to line.” Then using the method of application of areas where the chords of the curve parallel to the conjugate diameter became the ordinates to the principal diameter, he showed the curves had the property that the area of the “square on the ordinate” was either “deficient by,” “equal to,” or “exceeded by” when it was applied to the “parameter” line.
Having shown that “all sections cut in this manner” could be identified thusly, Apollonius simply said “let such a section be called” an ellipse, a parabola, or a hyperbola according as to its properties. Whether he simply dropped the “…-angled cone” designator because it was no longer applicable and reverted to a previously used descriptor or coined a new name for them is another of the many unknowns.
Regardless, Apollonius’ is credited with naming them.

jhmc 2014-10-28