A simple explanation of Archytas' Cube Duplication Solution

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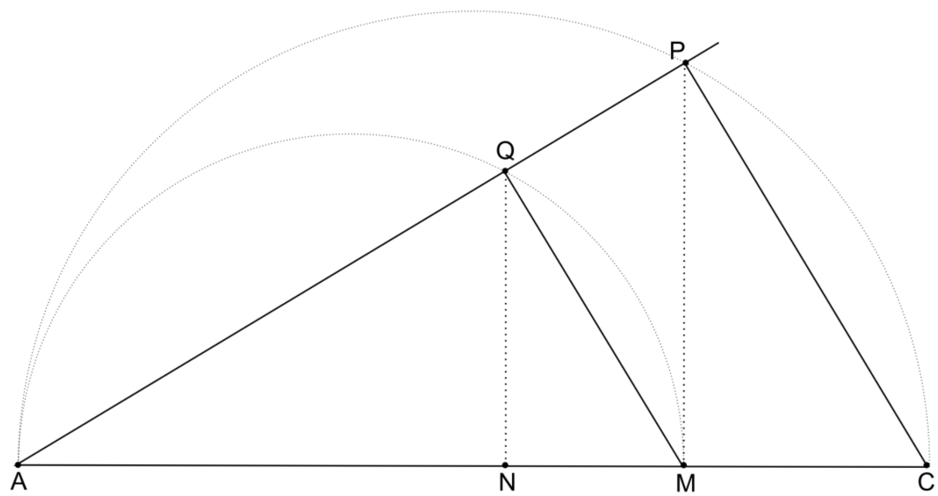
It is not known how Archytas conceived or constructed his solution to the duplication of the cube problem. In this paper we follow a non-embellished descriptions of the method as per Knorr and show that it is a clever 3D implementation of a 2D method of finding two means between two extremes.

Constructing two means between two extremes is not difficult. Draw a line AC as shown in this drawing and construct a semicircle with AC as diameter. Then draw a second line from A not collinear with AC to intersect the semicircle at P.



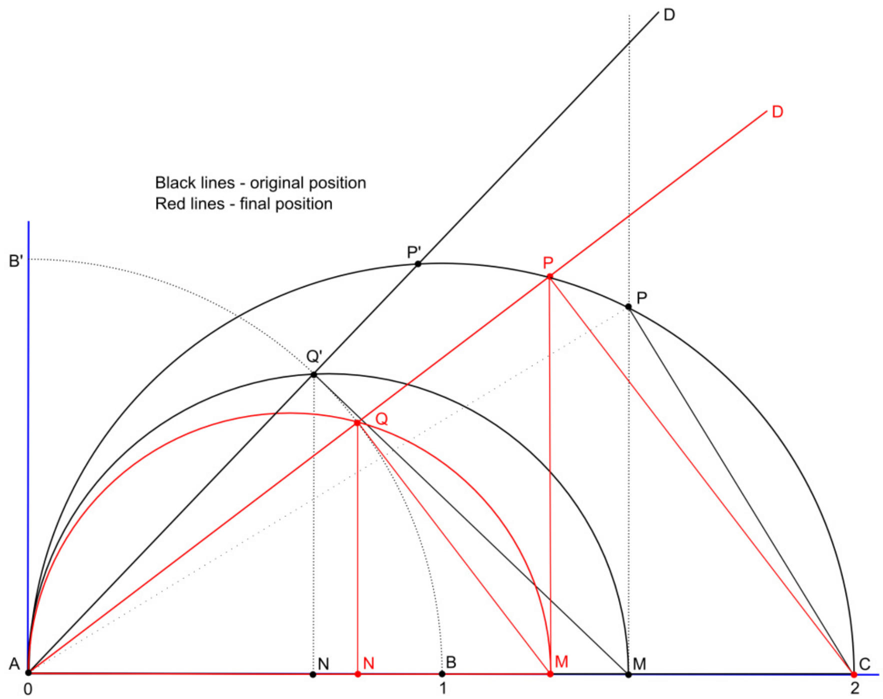
From P drop a perpendicular to AC intersecting it at M and then draw a second semicircle with AM as diameter intersecting AP at Q. Angles AQM, APC being included angles of semicircles are right angles as are angles AMP and ANQ which are base angles of perpendiculars.

The three triangles AQM, APC, and AMP are right triangles and in addition have the angle QAM is common which makes them similar triangles. Corresponding sides of similar triangles are proportional and we can now write: AC:AP = AP:AM = AM:AQ which is the basis for saying that AM and AP are two means between and.

The Greeks compounded this but we simply treat the ratios as fractions to arrive at $\frac{AM^3}{AQ^3} = \frac{AC}{AQ}$. Thus, if AQ is the length of the side of the original cube, a cube constructed with sides of length AM will have the volume AC:AQ. If we construct the figure so that AQ =1 and AC = 2 the equation reduces to AM=$\sqrt[3]{2}$.

There is no magic here but the construction is deceptively simple. The difficulty comes when both AQ and AC are given and some method of finding the correct angle QAM must be found in order to complete the construction. One such method is shown here in a figure taken from an interactive GeoGebra 2D model. Unfortunately the Greek geometers did not have such available and would have had to resort to mechanical devices. The GeoGebra model is discussed first and then a mechanical model is presented that the author believes accurately reflects Archytas' clever implementation of a 3D method for manipulating the 2D method used in the GeoGebra model. Henceforth we refer to this 2D model as the GGB model.

The GGB model is based on the ideas presented above for the specific problem of finding two means between two lines of length 1and 2 respectively. This is done by constructing line AC with a length of 2 and constructing a semicircle with AC as its diameter. The quadrant arc ABB’ is then constructed with point A as its center and a radius half the length of AC. Line AD is then drawn intersecting semicircle AC at P’ and ABB’ at Q’. Q’ is constrained to move along the quadrant to insure that the length of AQ’ is always 1 as the line AD is rotated about A.



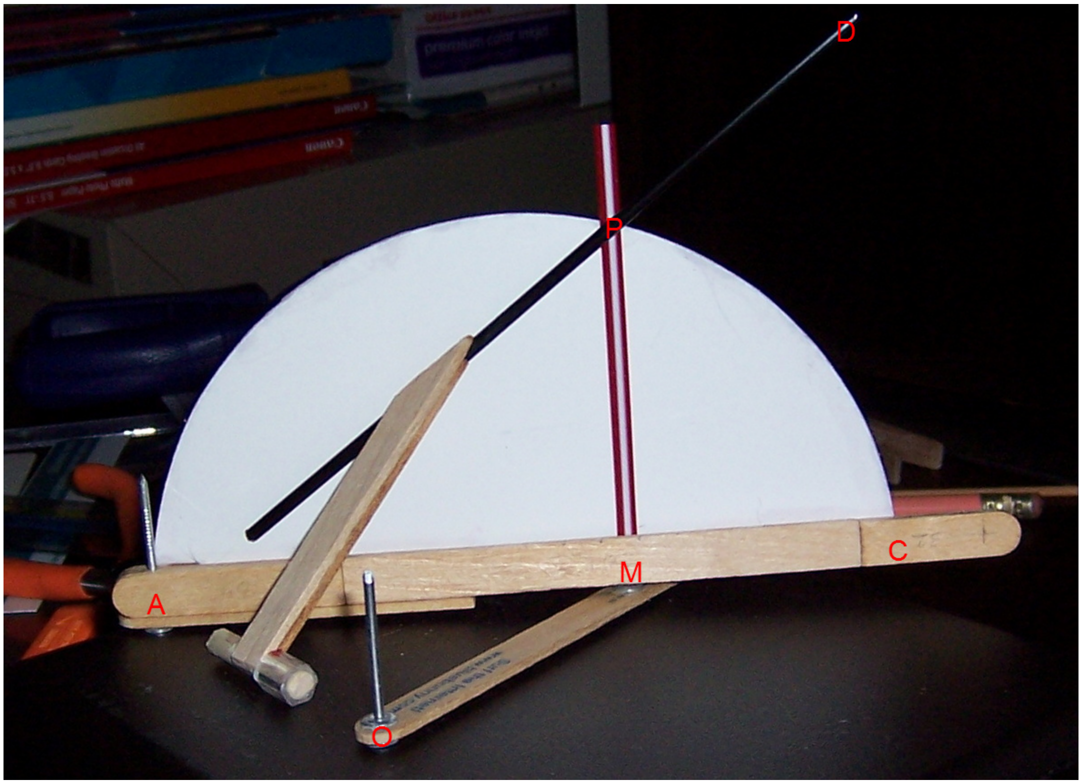
Semicircle AQ’M is constructed with its diameter collinear with AC and passing through points A and Q’. This is where GGB does for us what is not easily done mechanically. It allows Q’M to be constructed so that angle AQ’M is always a right angle as point Q’ is moved by adjusting the position of point M on the line AC.

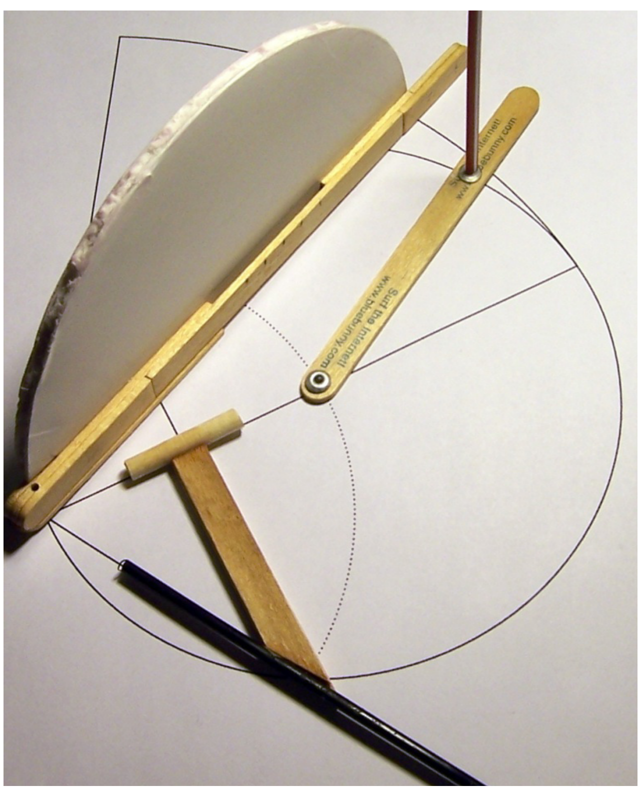
Finally, PM is constructed perpendicular to AC through M intersecting the semicircle AC at P. P lying on the semicircle insures that angle APC is always a right angle as point P is moved.

The model thus initially looks like the black figure and has to manipulated to produce the required results. This is done by dragging D to rotate AD clockwise about A. This moves point P’ to the right and also moves point Q’ downward. Moving Q’ downward reduces the diameter of the semicircle AQ’M moving point M to the left and as M moves it moves P to the left. Such is the magic of GGB.

With continued dragging of point D, points P’ and P can be made to coincide at P and the other parts of the model will then be in the positions shown by the **red lines** and **points**. When this configuration is reached, the triangles constructed (**red lines**) will be similar right triangles and their sides will be in continued proportions as in the example above. Thus AC:AP = AP:AM = AM:AQ from which $(\frac{AM}{AQ})^3 = \frac{AC}{AQ}$. But by construction AQ =1 and AC =2. Thus AM = $\sqrt[3]{2}$ and is the required line.

Obviously Archytas did not have access to modern computers and had to implement any method that was not purely geometrical with a physical mechanical model. While the model described here is constructed from pop rivets, popsicle sticks, coffee stirrers, and krazy purple glue, it is believed that a competent craftsman of that time could have constructed a working implementation. It is our interpretation of Archytas' remarkably clever implementation of a 3D method to manipulate the simple 2D method that GGB model was based on.



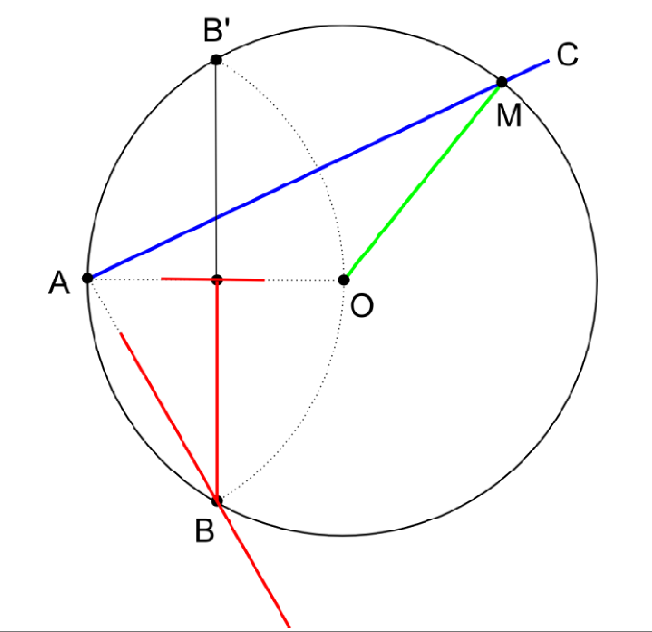


Our model consists of a base circle and three other parts. The base circle is used for layout, as a reference plane and for proof of the construction. The other three components shown assembled in the previous picture and separated in the figure to the left are:

(1) A semicircle of diameter AC mounted perpendicular to the base circle plane in such a manner that it pivots about point A on the base circle.

(2) A perpendicular bar MP mounted on a base arm OM that it pivots about the center of the base circle so that M is always located on the base circle.

MP passes through a slot in the base of the semicircle linking them so that moving one also moves the other one. As the semicircle rotates MP acts as a line perpendicular to the base on the base circle with P lying on the perimeter of the semicircle.

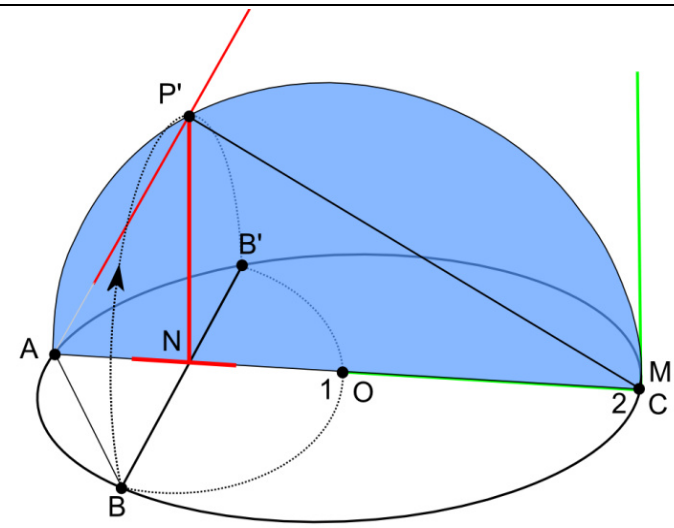


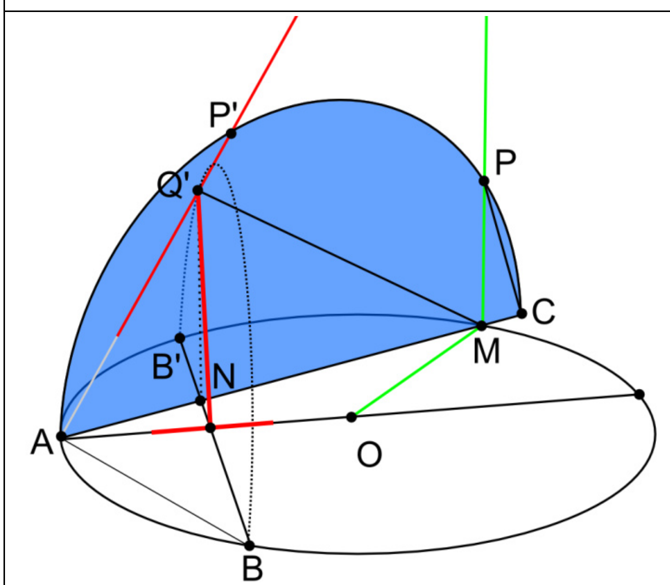
(3) An angle bar, OAD. The figure to the left shows the base circle used to layout the angle bar. An arc with radius half that of the base circle and centered on A intersects the circle at B and B’. The angle bar is then the line OABD.

The bar is mounted on an arm that rotates about AO as shown in red. Mounting the bar in this manner allows the “angle” to be omitted to simplify construction while insuring that AB has a constant length of 1 as required.

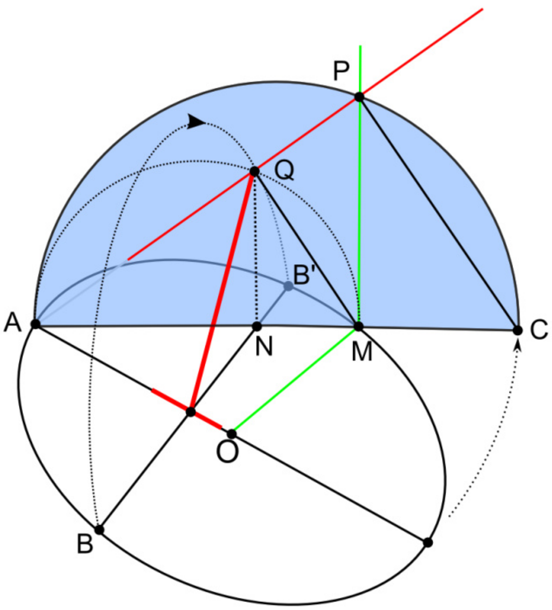
The blue line is the upright semicircle while OM is the base of the perpendicular bar, PM, that pivots about O.

To use the model, the angle bar is rotated about the axis AO in a counter clockwise direction (viewing AO from the A end). When it is rotated to the point that it touches the semicircle, NAP’ will lie on the plane of the semicircle. With further rotation the angle bar, still lying in the plane of the semicircle, will slide down its face and cause it to rotate about A. This in turn causes point M to move along the base circle.





With continue movement points P and P’ move toward each other along the perimeter of the semicircle as shown in the figure at the left. We note that P’ and Q’ are no longer the same point and the NQ’ is the altitude of AQ’M.



At some point the angle bar, the perpendicular bar and the top edge of the semicircle will coincide as shown in the earlier picture of the model and in the figure to the left. The triangles on the semicircle now appear similar to those in the 2D plane computer model. A circle AQM has been added to show that angle AQM is a right angle which requires further proof that is given below.

The semicircle BQB′ shows the path Q takes when the angle arm OAP rotates about AO. QN is thus an ordinate of the semicircle BQB′ and the mean of BN and NB′.

But AM and BB′ are intersecting chords of the base circle AC and hence QN = BN*NB′ = AN*NM meaning that QN is also the mean of AN and NM. AQM is thus a semicircle and AQM is a right angle as required. This use of the common ordinate of circles and/or ellipses was a common technique used in Greek geometry.

Angle APC is inscribed in the semicircle APC making it a right angle also. The triangles AQM, AMP, and APC are thus similar making their sides in continued proportion AC:AP = AP:AM = AM:AQ and as before this reduces to $(\frac{AM}{AQ})^3 = \frac{AC}{AQ}$. But by construction AQ =1 and AC =2. Thus AM = $\sqrt[3]{2}$ and is the required line.

In an Epigram addressed to Ptolemy, Eratosthenes proposes a simpler solution to the problem based on mechanical sliding panels to find the means. He criticizes Archytas’ solution as being difficult to construct and for his use of a cylinder.

PM in our model could be considered a cylinder generator just as the line AP could be considered a cone generator as Apollonius does later in his Conics. Eratosthenes does not mention “surfaces” and the mention of such relative to Archytas’ solution is first found in writings written several hundred years after the fact. Thus it seems highly unlikely that Archytas himself viewed this as a problem involving a cylinder, a cone and a torus as seems popular on today’s internet. I believe that rather than it being “a tour de force of spatial imagination” it is a clever 3D mechanical solution of a 2D problem by an accomplished applied geometrician.

January 27, 2014

Apr-01-2014 Complete revision to add pictures and explanatory text based upon feedback from a presentation made at ASMSA Mar- 20 2014.

July-20-2014 Minor correction to a sentence that was incomplete.