Cubes, Pyramids and Frustums

John H. McCoy

Cubes, like parabola, are similar and when you have seen one you have seen them all. They are also related in another way in that the use of a parabolic curve was first documented in Menaechmus' fourth century BC solution to the problem of doubling a cube.

Interest in the cube dates even further back in antiquity for Menaechmus' solution of finding two mean proportionals between two given lines is attributed to being first proposed by Hippocrates of Chios. Hippocrates was a merchant who, depending upon which story you believe, was bankrupted by either his own naiveté or by unscrupulous tax collectors and became a mathematician after moving to Athens. It is thought that he had likely been a student of Oenopides of Chios and, if he did not study with the Pythagoreans, he was at least influenced by them.

Hippocrates is known to have worked on the problem of doubling the cube but the original source of the problem remains unknown. The Egyptians were master pyramid builders and the problem of doubling the size of a pyramid is the same as that of doubling the cube raising the possibility that the problem may have originated there. It is known that he made trips to and studied in Alexandria.

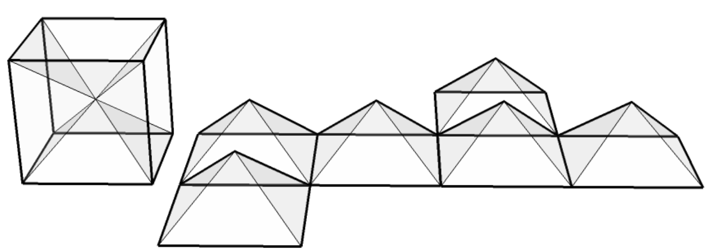
Finding two means between extremes is also tantamount to finding a cube root. Clay tablets on which the Babylonians inscribed, among other things, a table of cube roots for integers up to 30 have been found though the method used to arrive at the values is not known. Thus, through commerce, the Egyptians could have had access to the mathematics they needed to solve the "doubling" problem for pyramids but there is no evidence of them having done so.

Another possibility is that it was an intellectual problem that Hippocrates contrived based on something he observed being done mechanically. For instance, observing the casting of small but double or triple size pyramidions. Hippocrates may have first encountered the "cube" problem as a taunt of "let's see you do this with your geometry." This is of course all speculation and the problem may well have originated with Thales or Oenopides.

What's in a Cube

A cube can be decomposed into component parts of congruent solids. The most obvious are the two prisms that result from slicing a cube with a plane that passes through the diagonals of the faces of two opposite faces producing two triangular prisms each of which is half the volume of the cube.

The immediate interest here is in the six congruent square pyramids that result when, rather than slicing along one diagonal of a pair of opposite sides, four planes are used to slice along both diagonals of both pairs of



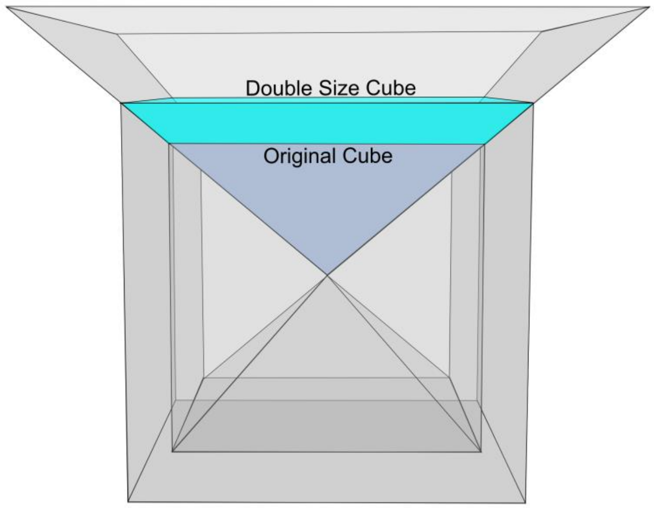
opposite sides. Each face of the cube becomes the base of a pyramid that has a height equal to one half the length of a side and a volume that is one sixth that of the cube. They are right pyramids (the apex is located over the center of the base) and have right apex angles. The drawing shows how the pyramids can be unfolded.

This establishes the connection between the "cube" and the "pyramid" problems. If we have a method to double the volume of a square base, right pyramid that has an apex angle that is right then we have a method to double the volume of a cube.

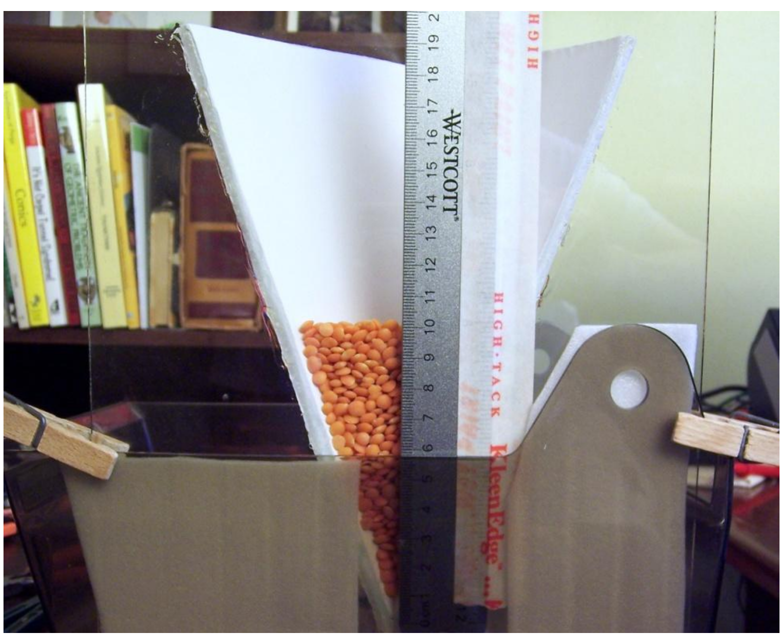
An Analog Computer

Assume a cube has been cut with four planes along the diagonals as just described. Produce the planes so as to extend the sides of the upper inverted pyramid as shown in the drawing below.

Fill the inverted pyramid with water - the dark blue fill - to the line that is the side of the original cube and which has been marked as 1 (one). Then add an equal amount of water - light blue fill - to double the amount of water. The new water line, marked 2 (two), will then be the required side length to double the volume of the cube. The wire frame outlines the required cube.



By adding the appropriate amount of water the size of the cube can be increased by any desired amount – tripled, quadrupled, etc. The ratio of the length of the raised water line to the length of the side of the original cube will equal the cube root of the ratio of the final water volume to the original volume. Thus, our device acts as an analog computer that can compute the cube root of any number within its capability.

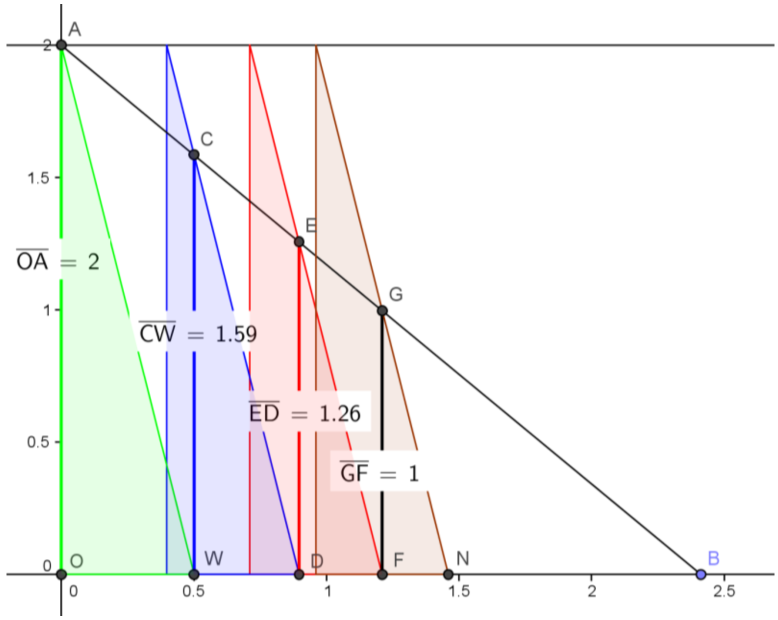


The picture to the left is of a model based on the above method. A square based pyramid has been cut in half along the diagonal of the base. One half has then been glued to a piece of glass and the apparatus mounted so that the glass is vertical.

The ratio of the depth of the total fill to the depth of the initial fill will be the cube root of the ratio of the volume of the total fill to that of the initial fill. The slope of the sides affects the amount of fill required but not the results.

Rather than water, red lentils were used for the fill. One can always cube the depth ratio and compare with the volume ratio. For a simple accuracy check use an initial fill of 1 unit and then add 1 additional unit of fill. This will give a result for a 2:1 fill ratio. Now add two more units of fill. The 4:2 depth ratio should be the same as that for the 2:1 while the 4:1 depth ratio should be the square of it.

It is interesting to demonstrate that whatever the depth of the initial fill, an additional amount that is seven times the initial volume is required to double the depth. i.e., a total fill of eight units. A different way of saying that seven eights of the volume of a pyramid lies between the base and the midpoint of its altitude.



Eratosthenes was not the first to solve the cube duplication problem but his solution, described as a gift to Ptolemy, may have been based upon the duplication of a pyramid describe above.

The figure to the left was taken from an interactive GeoGebra model of his solution method.

The four panels are identical except for color.

The position of the green panel is fixed while the other three are constrained to remain upright as they move along the X axis.

The line AB pivots about A. Points C, E and G are constrained to move along the sloping face of the panel on which the point is located as it moves vertically up and down along a line attached to the bottom edge of the adjacent panel to its left.

Using similar triangle it can be shown that the lines FG, DE, WC and OA are in continued proportion so that $\frac{OA}{CW} = \frac{CW}{ED} = \frac{ED}{GF}$ from which $\left(\frac{ED}{GF}\right)^3 = \frac{OA}{CW} * \frac{CW}{ED} * \frac{ED}{GF} = \frac{OA}{GF} = \frac{2}{1}$ for the configuration as shown. Taking cube root of both sides of the equation it becomes $\frac{ED}{GF} = \sqrt[3]{2}$.

Although Eratosthenes reportedly constructed a panel device that could find any number of means between a range of extremes, but no such device has yet been found. In view of the rather recent discovery of the Antikythera Mechanism that such a device did exist cannot be dismissed out of hand.

Returning to Hippocrates.

Did the Egyptians have such a device as the water calculator? Probably not and almost certainly not in the form shown above. Hippocrates' achievements though attest that he was an astute observer and it is plausible that if some such device in any form existed, and he occasioned to observe it, his analysis of its operation could have suggested the two means between extremes solution. Again though, the only thing known for sure about the "cube" problem is that he is credited with suggesting such a solution.

Slightly more is known about the calculation of the volume of a pyramid, or rather, the volume of the frustum of a pyramid because a formula for calculating the volume has been found on an Egyptian papyrus dating back to 1850 BC.

However, like the Babylonian cube root table, how the Egyptians arrived at the formula remains a mystery. It is interesting to note that in the book "*Great Moments in Mathematics before 1650,*" Howard Whitley contrasts the correct formula used by the Egyptians with an incorrect one used by the Babylonians. Perhaps the Babylonians can be excused since they were not big time Pyramid builders, as were the Egyptians, and may never have suffered the consequences of coming up short on materials in a construction project.

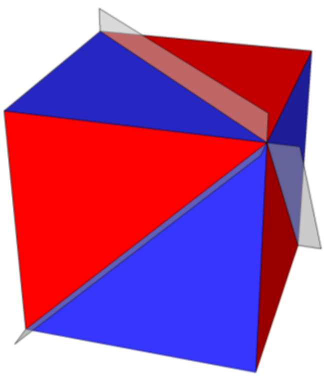
Heron of Alexandria is credited with giving the oldest documented (1^{st} century AD) derivation of the formula which multiplies the height by what is now known as the Heronian mean. I have not seen his derivation but most references seem to imply that he was interested in various means between two numbers and not specifically interested in frustums.

Most modern derivations use calculus and or extensive algebraic manipulations that some think are beyond the capabilities of Egyptian mathematics of that time period. Even the derivations that are geometrically based generally assume that the Egyptians knew a method of calculating the volume of a pyramid. While it is reasonable to assume this, it is not known that they did and a method that does not require such an assumption nor require algebraic manipulations is presented here.

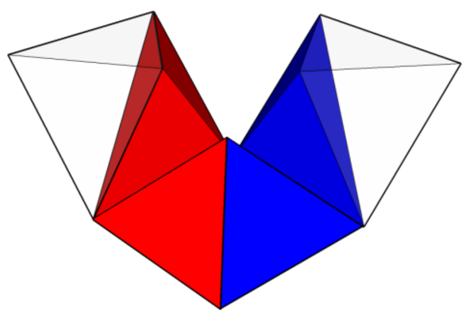
A Different Cut

We again cut a cube into six pyramids of equal volume but this time three planes that intersect collinearly with a diagonal of the cube are used and the resulting pyramids have triangular bases.

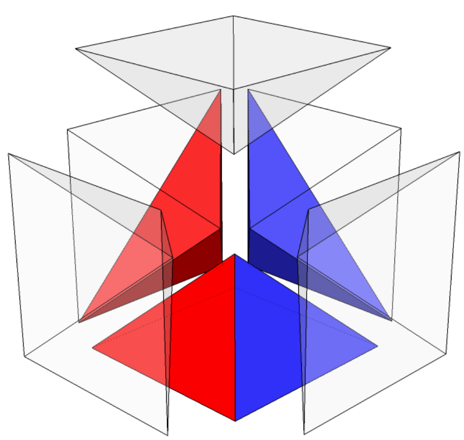
The next drawing shows the three planes cutting the cube along the diagonals of the front, top and right sides. The pyramids discussed previously were square pyramids and the base of each was a face of the cube. This time



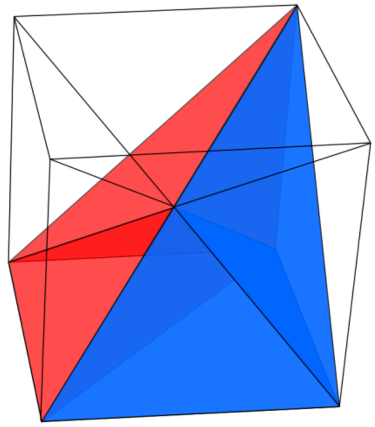
the three planes essentially cut the square base of each of those pyramids along a diagonal while leaving an uncut edge between it and the half of a pyramid on an adjacent face with which it is combined to form a triangular pyramid.



The drawing above shows this in more detail. In particular it shows that the pyramid on the bottom face has been cut along a diagonal into red and blue halves and the side faces are folded along the uncut edge to show how those pyramids on the adjacent faces have also been cut and how they fit with the parts of the pyramid on the bottom face of the cube to form the triangular pyramids. (In the next photo the fold lines have been cut)



The red and blue triangular pyramids thus formed are mirror images of each other and are said to be indirectly congruent. Four other triangular pyramids are also in the cube. Two will be congruent with the red one and two will be congruent with the blue one.



This last drawing shows two triangular pyramids combined to form a square pyramid with the bottom face of the cube as its base and an upper corner of the cube as it apex. Now we have two types of square pyramids to deal with. The smaller right ones and the oblique corner ones.

Each triangular pyramid has the same volume as a right one while each corner pyramid being composed of two triangular ones has twice the volume. Since each small right pyramid is one sixth the volume of the cube the corner pyramid has a volume that is $2*(1/6) = 1/3$ the volume of the cube.

If **A** is the area of a face, and **H** is the height of a side of the cube, then the right pyramid on the base of the cube has a base area of **A** and height of $\mathbf{h} = \mathbf{H}/2$ while the corner pyramid has the same base and base area of **A** but a height of $\mathbf{h} = \mathbf{H}$.

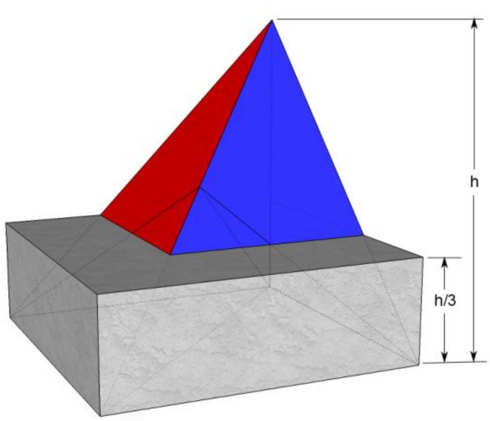
For the symmetrical pyramid then: $\mathbf{V} = \mathbf{HA}/6 = (\mathbf{H}/2)\mathbf{A}/3 = \mathbf{hA}/3$.

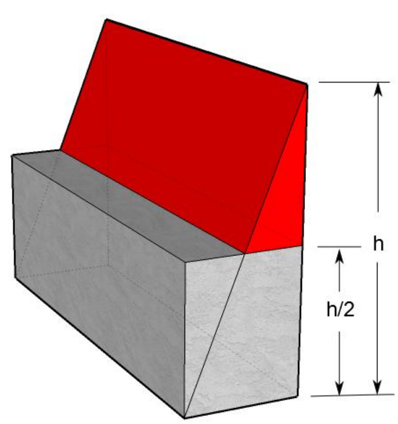
For the second: $\mathbf{V} = \mathbf{HA}/3 = \mathbf{hA}/3$.

Thus deriving the general formula for the volume of a pyramid $\mathbf{V} = \mathbf{hA}/3$ Where **h** is the height of the pyramid and **A** is its base area.

A Different Way of Looking At It

Most often we want to plug some values into the volume formula and crank out the volume. But another way of looking at it is, that if we keep the base area, A, the same the equivalent volume of the pyramid is a square prism 1/3 the height of the pyramid.



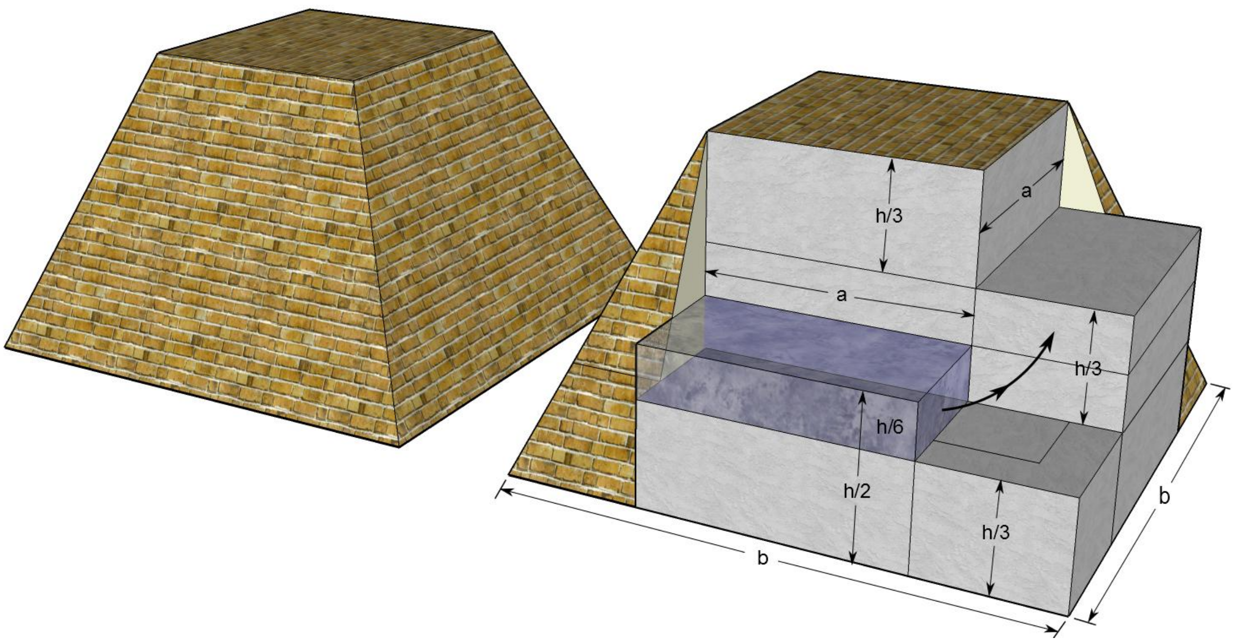


Similarly a right triangular prism can easily be converted into an equivalent rectangular prism having the cross section area of the triangular one. This means that if one side of the triangle is the base and the other is the altitude then the equivalent volume of the triangular prism will be a rectangular prism with the same base but 1/2 its height.

Finding the Volume of a Frustum

Using the above ideas we transform a square frustum with top and base side lengths of a and b into rectangular blocks having the same aggregate volume which can be easily calculated. For this purpose we consider the frustum shown below and treat it as being composed of a central rectangular core having sides of length **a** and height **h**. The sides we treat as being composed of right pyramids at each corner of height **h** that are filled in between with right triangular prisms having an altitude of **h**.

The figure shows how this can be done. First transform a corner pyramid in to a block having the same base as that pyramid but one third its height (**h/3**). Then the two adjacent sloping walls which are the same length as



the sides of the top and have the same base width as the corners are transformed into blocks that have a height one half the original height of the wall (**h/2**).

The wall blocks are thus one sixth of the height of the frustum higher than the corner block. Take this excess from one side and place it on top of the other transformed side as shown by the arrow.

Now apply the same transformation to the other three corners and two sloping walls to transform the frustum into a base layer that has an area of **b*b**, a middle layer with an area of **a*b** and a top layer of **a*a**.

Each layer has a depth of one third the height of the frustum so the total volume of the layers and hence of the frustum is:

$$V = h(a^2 + ab + b^2)/3$$

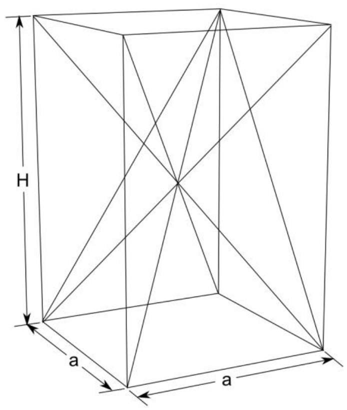
which is the formula found in problem 14 of the 1850 BC Moscow papyrus.

Again, no one knows how the Egyptians came up with their formula but given their expertise in pyramid building, it certainly appears easily within their mathematical capabilities to have done so. The fact that they got it right while the Babylonians didn't also speaks highly of their expertise.

Appendix

Stretched Cube

In this wire frame drawing drawing a cube with sides of length **a,** faces of area $\mathbf{A}_C = \mathbf{a*a}$ and a volume of $\mathbf{V}_C = \mathbf{a}^3$ has been cut with four planes as before and stretched to a height of **H** while the top and bottom faces are unchanged.



The six facepyramids in the cube were all congruent and each had a volume of one sixth of the cube, $\mathbf{V}_P = \mathbf{V}_C/6 = \mathbf{a}^3/6$. In a prism however, the four right pyramids on the side faces are congruent but they are no longer congruent with the two on the top and bottom faces.

Assume the cube has been stretched so that the height of the resulting prism is double the length of a side of the cube: $\mathbf{H} = 2\mathbf{a}$; then its volume $\mathbf{V}_{PRISM} = \mathbf{H*a*a} = 2\mathbf{a*a*a} = 2\mathbf{a}^3$ is twice that of the cube.

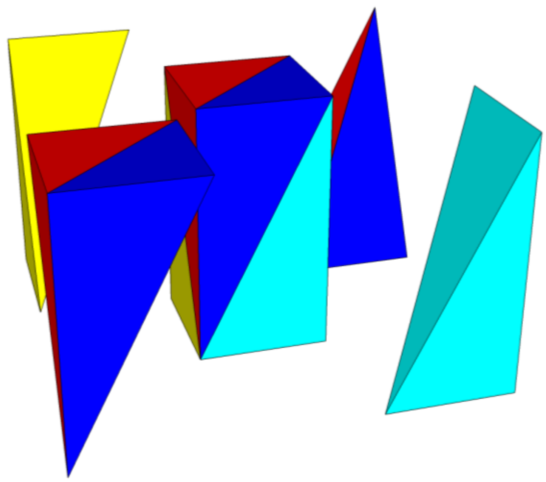
Now, the base area of the pyramids on the top and bottom faces is unchanged at $\mathbf{A} = \mathbf{a*a} = \mathbf{A}_C$, but their height $\mathbf{h} = \mathbf{H}/2 = 2\mathbf{a}/2 = \mathbf{a}$ has been doubled. As was previously shown, doubling the height of a pyramid without changing its base area doubles the volume of the pyramid. Thus the volume of the pyramids on the top and bottom faces of the stretched cube will be twice the volume of one of the face pyramids in the cube, $\mathbf{V}_{CF}$. That is $\mathbf{V}_{T\&B} = 2\mathbf{V}_{CF} = 2(\mathbf{a}^3/6) = \mathbf{a}^3/3$.

Knowing now the volume of the prism and the volume of the pyramids on the top and bottom faces we can calculate the total volume of the four pyramids on the side faces of the prism.

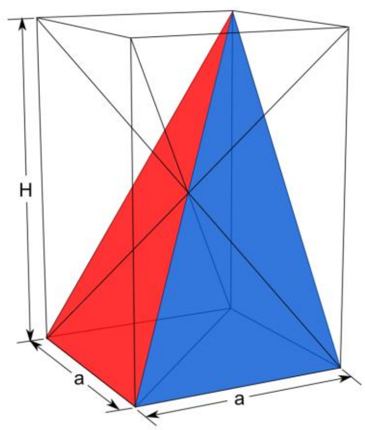
$$\mathbf{V}_4 = \mathbf{V}_{PRISM} - 2\mathbf{V}_{T\&B} = 2\mathbf{a}^3 - 2\mathbf{a}^3/3 = 4\mathbf{a}^3/3 = 4\mathbf{V}_{T\&B}$$

The volume of each side face pyramid is one fourth this and, thus as we assumed, all six face pyramids are shown to be of equal volume just as they were in the cube even though they are not all congruent.

Now consider how the stretching affects the triangular pyramids created when the cutting is by three planes rather than four. The base of one pair of red and blue ones share the top face of the stretched cube and the other pair share the bottom face. The remaining yellow and cyan triangular pyramids however, are composed of halves of two side face pyramids. They thus have the same volumes as the red and blue ones and are mirror images of each other. Unlike in the cube, however, they are not congruent with either the red or blue ones.



Recall that each of the triangular pyramids is comprised of halves of face pyramids which are now not all congruent thought they are still of equal volume. Consider again the red and blue ones that share the bottom of the square prism (stretched cube) and form a square corner pyramid.



Both are composed of half of the square pyramid on the bottom of the prism plus half of a rectangular pyramid on a side face. They are thus of equal volume and are again mirror images, or, indirectly congruent as before.

The same analysis will show that the two triangular pyramids whose bases share the top of the prism are also mirror images and indirectly congruent . The red ones are congruent as are the blue ones.

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