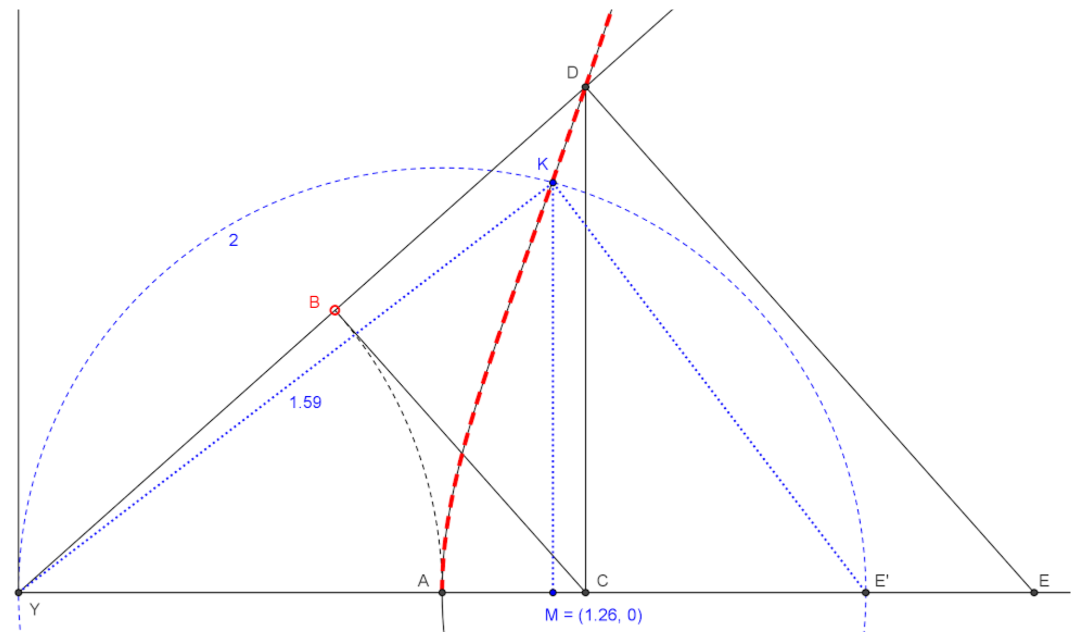
Eudoxus

Eratosthenes referred to Eudoxus' solution to the cube duplication problem as the "curves of that God fearing Eudoxus." No one knows what exactly these curves were but one conjecture is that his solution was based upon the projection onto a plane of a supposed Archytas 3D solution to draw a 2D curve now called Eudoxus' Kampyle. The Kampyle can be described by $y^2 = x^4 - x^2$. in today's Cartesian coordinates.

Heath objects to that idea in itself in that Eudoxus was a premiere mathematician and would have considered such to be unethical. A much simpler method of drawing the Kampyle is presented here that uses a device based on the same underlying similar right triangles used by Archytas but Eudoxus uses them in an entirely different manner.



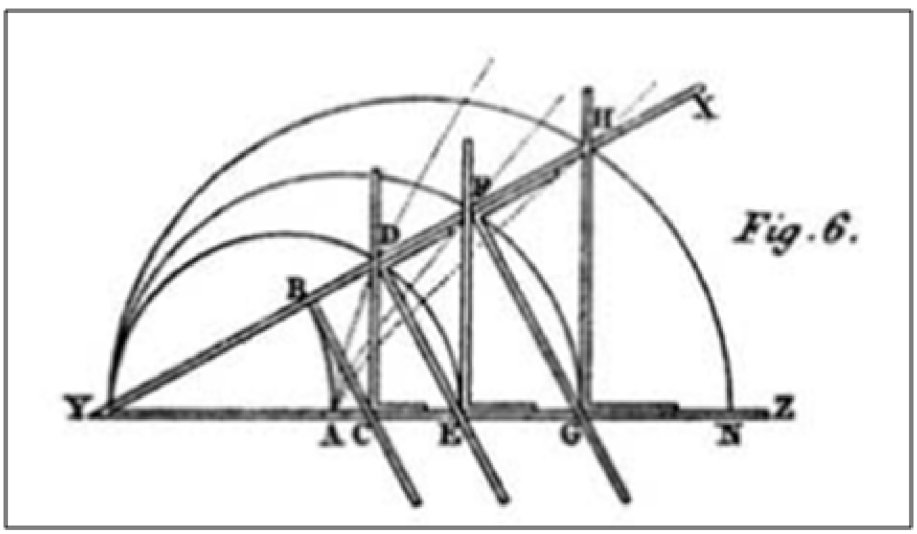
ABC, ACD, and ADE are those similar right triangles and YC and YD are the two means between YB and YE. YBD is moved by dragging B which is constrained to along the arc.YAB. YB=YA=1. Dragging B clockwise causes point C and the perpendicular CD to move to the left. This moves point D and, through DE, point E to the left. If B is dragged until YE = 2, YC will be $\sqrt[3]{2}$. However, this is not thought to be the way Eudoxus did it. Rather he likely traced the locus of point D to produce the dashed red curve by dragging point B.

Each point on this curve represents a unique combination of YD, YC and YE where YD and YC correspond to the two means between 1 and the upper extreme YE. Once the trace of D is drawn, the two means between 1 any given number can be found by drawing a circle having a diameter YE' equal to the given number and lying along YE. The circle will intersect the curve at some point K. YK is then one mean and YM, the projection of YK onto YE, is the other of two means between 1 and the diameter of the circle YE'. This is illustrated with the blue circle which was drawn with a diameter of 2 and $YM = \sqrt[3]{2} = 1,26$ while $YK = \sqrt[3]{2^2} = \sqrt[3]{4} = 1.59$.

The dashed red curve lies atop a black plot of the equation of the Kampyle given above, $y^2 = x^4 - x^2$. This was done to show that, although both solutions make use of the continuing proportions of similar right triangles in a plane, Eudoxus' Kampyle is easily drawn without reference to any supposed solution by Archytas.

Descartes' Mesolabe

The knowledge about Eudoxus' solution comes from Eratosthenes' epigram where he describes it as *"that shape which is curved in the lines That Divine Eudoxus constructed."* While the device suggested above used only geometry appropriate for the time to produce a curve that is a legitimate solution, that Eudoxus used such a device remains a conjecture. Before rejecting it out of hand, however, consider the device shown in this figure that was created some 2 millennia later by Descartes.

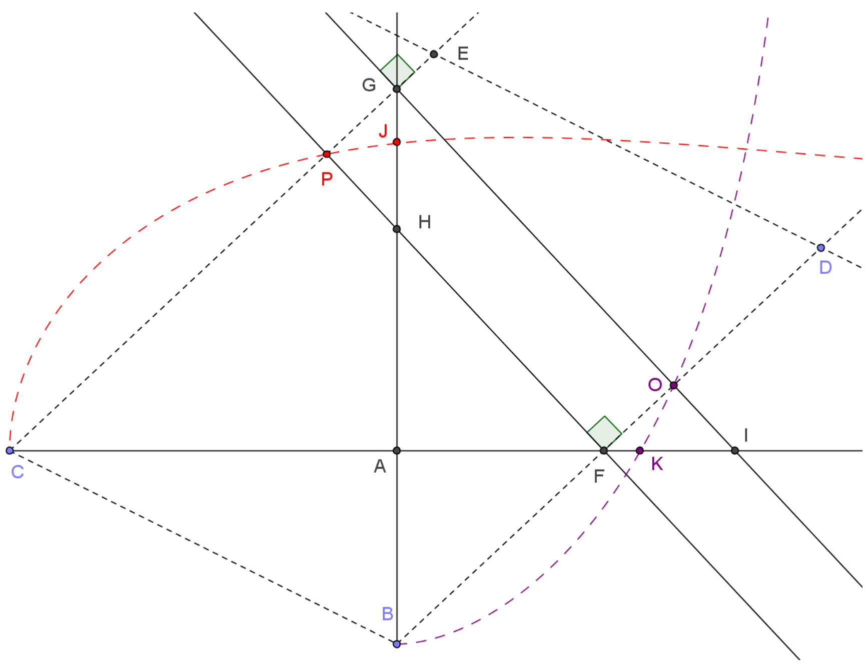


Both the drawing and the following description of its use are from footnote 71, pgs. 119-120 of "Descartes' Mathematical Thought" by C. Sasaki.
"For if we wish to find two mean proportionals between YA and YE, we have only to describe a circle whose diameter is YE; and because this circle cuts the curve AD at point D, YD is one of the required mean proportionals. The demonstration is obvious to the eye, merely by applying this instrument to the line YD: for, as YA, or YB which is equal to it, is to YC, so YC is to YD, and YD to YE."

Is the similarity of Descartes' mesolabe to Eudoxus' device a mere coincidence? Perhaps, as there is nothing to indicate that Descartes was aware of Eudoxus' work. It should also be remembered that it is only a conjecture that Eudoxus' solution was the curve known as the Kampyle.

On the other hand Descartes is known to have studied Eratosthenes' compass mesolabe and even used the same name to describe his device. However, Eratosthenes' device used only a single leg of sliding right angles whereas Descartes' uses two. Thus, while Descartes' device is not a copy, studying Eratosthenes' device may have been the catalyst Descartes needed.

There is also however, another possibility. Eudoxus was the "middle man" of the three credited with first solving the problem of doubling the cube. Thus it is conceivable that he chose an entirely different basis for his solution than what Archytas used and that rather than a Kampyle curve his solution was "…that shape curved in the lines…" as alluded to by Eratosthenes. We present such a solution here but defer its discussion it until after a discussion of "Plato's Frame."



$\frac{CA}{BA} = \frac{2}{1}$ and CA is orthogonal to BA. BCED is a parallelogram. The red and maroon curves are plots of the points P and O as D is moved rotating the parallelogram about side BC. The red curve intersects BA extended at J and the maroon curve intersect CA extended at K.

It can be shown that $\frac{CA}{JA} = \frac{JA}{KA} = \frac{KA}{BA}$ from which it is deduced that JA and KA are two means between 2 and 1.

Either method suggested here would seem to be consistent with Plutarch's contention that the methods of Archytas and Eudoxus were instrumental rather than rigorously geometric.

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