The Egyptian Connection

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Our understanding of ancient Egyptian mathematics is based upon a relatively small collection of artifacts. Mostly these are papyrus fragments recovered in the 19th century AD that were produced in the early 2nd millennium BC by copying from even earlier sources. They contain some tables useful for making calculations and examples of reckoning solutions to problems such as would be needed for governing, commerce and agriculture.

Written in hieratic script and hieroglyphs there are no “formula” as such. Rather, the examples are mostly solutions where some numerical and/or geometric data is given along with a sequence of steps that, when performed, results in a “correct” result. What the result represents may have to be inferred from the context.

It is this algorithmic processing of specific data that various researchers have analyzed and translated into formula that to evaluate requires performing a sequence of operations mimicing those in the manuscript. As would be expected, differences of opinion regarding both the translations and their interpretation exist. Consequently Egyptian mathematics tends to be defined more by conjecture and what’s missing from the record than from direct evidence.

There is, for example, no direct evidence that the Egyptians knew the very simple calculation required to reckon the volume of a pyramid though it is generally accepted that they did. Only when an example from that era is found will that question be answered definitively.

Lacking such, however, consider problem 14 from the Moscow papyrus which correctly reckons a value 56 and assures you that it is the correct answer.

If someone says to you:
"A pyramid of 6 for the height by 4 on the base by 2 on the top."
You are to square this 4, the result is 16.
You are to double 4 the result is 8.
You are to square this 2, the result is 4.
You are to add the 16 and the 8 and the 4 the result is 28.
You are to take 1/3 of 6, the result is 2.
You are to take 28 two times, the result is 56.
Behold, 56 you will find is correct.

The question the translators had to answer is "if 56 is the answer, what was the question?" To determine that, the translators observed that the same sequence of steps would be taken to evaluate the expression $(4^2 + 2 * 4 + 2^2)(6/3)$.

A crude line drawing accompanying the problem suggested that the problem could pertain to a pyramid frustum and the given height, base and top values were plugged into the equation for the volume of a pyramid frustum, $V_f = (b^2 + a * b + a^2)(h/3)$ giving $(4^2 + 2 * 4 + 2^2)(6/3)$. From this the translators deduced that problem 14 calculated the volume of a pyramid frustum providing direct evidence of the Egyptians' ability to reckon the volume of a frustum.

There is, however, no such direct evidence that the Egyptians knew how to reckon the volume of a full pyramid. That they could devise a method of reckoning the volume of the frustum without knowing how to do so for a full pyramid seems unlikely. The usual suggestions for deriving the frustum formula require algebraic skills more advanced than those assumed possessed by the Egyptians of that time resulting in a conundrum that continues to baffle geometers.

Here it is assumed that the Egyptians did indeed know that the volume of a pyramid of height h and a square base with sides of length s is given by the formula $V = \frac{h}{3}s^2$. The validity of this assumption will be revisited after showing that when this full pyramid formula is known, deducing the frustum formula requires little more than decomposing 7 into 4+2+1 and would have been well within the Egyptian geometers' capabilities.

Begin with a pyramid of height h_a that has a base with sides of length a. Extend the faces so that the height of the extended pyramid becomes $h_b = 2h_a$. The Egyptians' were familiar with "bank" and "batter" calculations and could have determined that the extended pyramid is related to the original one by $\frac{b}{a} = \frac{h_b}{h_a} = 2$ from which the sides of the extended pyramid base $b = 2a$.

The original pyramid is the upper part of the extended one and has a volume $V_u = a^2\frac{h_a}{3}$ while the extended pyramid has a volume $V = \frac{h_b}{3}b^2 = \frac{2h_a}{3}(2a)^2 = 8a^2\frac{h_a}{3} = 8V_u$. The lower portion of the extended pyramid that remains when the upper part is removed forms a frustum of height $h_f = h_b - h_a = h_a$ and its volume can be obtained by subtracting upper volume V_u from the extended volume V giving

$V_f = V - V_u = 8V_u - V_u = 7V_u = 7a^2\frac{h_a}{3} = ((2a)^2 + (2a)a + a^2)\frac{h_a}{3}$.

Replacing b with $2a$ leads to the result $V_f = (b^2 + ab + a^2)\frac{h_f}{3}$ which is the algebraic equivalent of the method in the Moscow papyrus.

The Egyptians were pragmatic and if this met their needs they likely would not have carried it any further. However, with a little algebra it is not difficult to show a general solution by extending the original pyramid so that $h_b = nh_a$ which also makes $b = \dfrac{h_b}{h_a}a = na$. Then following the pattern found in the previous analysis.

$h_f = h_b - h_a = nh_a - h_a = (n-1)h_a$ from which $h_a = \frac{h_f}{(n-1)}$. Finally then,

$V_f = (n^3 - 1)V_u = (n^3 - 1)\frac{h_a}{3}a^2 = \frac{(n^3-1)}{(n-1)}\frac{h_f}{3}a^2$

$= (n^2 + n + 1)a^2\frac{h_f}{3} = ((na)^2 + (na)a + a^2)\frac{h_f}{3} = (b^2 + ab + b^2)\frac{h_f}{3}$ as before.

The Ahmes papyrus is named after the scribe who produced it by copying an existing manuscript that was several centuries old . The problem selection and orderly arrangement are likely the same as that from which it was copied. Interestingly there is not a problem involving the volume of a pyramid but there are several problems involving the seked about which more will be said later.

It has been noted that by setting a to zero in the frustum formula that the volume reckoned will be that of a full pyramid having a height h. Zero was not a number with which the Egyptians were familiar but from both observation and their use of the seked, they would know that the top dimension became smaller as the height approached that of a full pyramid. Thus, if the Egyptian algorithm skipped the three steps of the procedure shown in red when there was no top dimension, it could be used to correctly reckon the volume of a full pyramid as well as that of a frustum yet not directly involve the need to use zero..

In contrast to the Ahmes manuscript, the Moscow papyrus was prepared by an unnamed scribe. The problem selection and organization are rather random which some say indicate the author was a scribe in training and the material copied was not necessarily selected for its usefulness to the reader. If only a problem involving a full pyramid had been included then the absence of one involving a frustum might justify assuming that the frustum formula was unknown. Apparently then it is only by chance that a frustum problem was included and led to the discovery that the Egyptians could correctly reckon the volume of a frustum. But their use of the frustum formula makes it difficult to dismiss that they knew the full pyramid formula.

As little as is known about Egyptian mathematics in Ahmes' time, even less seems to be known about its status at the time Thales is said by Eudemus to have studied with the Egyptians and brought their knowledge of geometry to Greece and added many things. Several historians who have researched and written on the subject make note of the paradox that more detailed information pertaining to early greek geometers comes from authors writing a millennium or more after the fact than comes from the geometer's own writings or those of their contemporaries.

Thales was from Miletus and after returning from Egypt he is said by Eudemus to have developed a geometrical method for determining the distance a ship was from shore using similar triangles and proportions. This is consistent with the story that while in Egypt he had measured the height of one or more pyramids by comparing the shadow of the pyramid with that of a gnomon. The Egyptians are known to have been using the properties of similar right triangles in the time of Ahmes and Plutarch in the Banquet of the Seven Wise Men tells us that the king of Egypt admired Thales

> *. . .and particularly liked the manner in which you [Thales] with little labor and no help of any mathematical instrument you took so truly the height of one of the pyramids; for fixing your staff erect at the point of the shadow which the pyramid cast, two triangles being thus made by the tangent rays of the sun, you demonstrated that what proportion one shadow had to the other, such the pyramid bore to the stick*.

Egyptians measured the batter (inward slope) of pyramid faces by their deviation from vertical per cubit rise from the base. This is usually glossed over as just being what we now call the cotangent of the angle the base makes with the face and treated as a "nothing new here, move on" event. However, the Egyptians had no notion of the measure of an angle hence no trigonometry. This is more aptly described as an early use of the properties of similar triangles where the given triangle was a right triangle having a side of unit length. The intercept theorem attributed to Thales would seem to be a generalization of the Egyptian's use of the seked.

Given that pyramids typically have a height greater than half their base the run per cubit rise would always be less than 1. For example a 100 cubit high pyramid having a base of 100 cubits would have a run to rise ratio of $\frac{run}{rise} = \frac{50}{100} = 0.5$. The fractional value was used for reckoning but converted to and referenced in seked which is the ratio expressed in the fractional parts of a cubit (i.e., 7 palms per cubit and 4 fingers per palm). Thus, for our example, the run to rise ratio would be 3½ palms or 3 palms 2 fingers.

In problem 7 of the Moscow papyrus dealing with plane geometric figures the bank rather than the seked was used as the unit of measure. The bank is defined there as a dimensionless value equal the ratio of rise to run both measured in the same units.

In the preface to the "Quadrature of the Parabola" Archimedes credited Eudoxus with discovering the theorem "the volume of a pyramid (or, cone) is one third the volume of a prism (or, cylinder) on the same base and having the same height." Later in the preface to "The Method" he amended his claim to acknowledge that "this was first put forward by Democritus and later proven by Eudoxus."

Archimedes addressed The Method to his friend Eratosthenes and indicated it covered a method used to prove theorems Archimedes had previously sent to him. In that previous communication he had mentioned Eudoxus as the first to discover the volume of a prism theorem. His casual mention of Democritus later in The Method seems to suggest he assumed that Eratosthenes was in the know about Democritus.

An assumption that seems reasonably justified. Eratosthenes was called a man of learning, was a Mathematician, Astronomer, Geographer, Musician, as well as a Historian. As chief librarian of the Alexandria library Eratosthenes had access to resources not readily available to everyone and likely knew details about Thales, Anaximander, Democritus, Hippocrates and others from the school at Miletus many of whom both studied in Egypt and had common contact with the Pythagoreans and other schools in Ionia.

Democritus was an atomist who lived during the time of the early years of Greek mathematics. In addition to the theorems on the volumes of pyramids and cones he is also noted for a paradox involving the slicing of a pyramid that he proposed to counter to some of Zeno's paradoxes. About himself he is supposed to have said:

> *"I have wandered over a larger part of the Earth than any other man of my time, inquiring about things most remote; I have observed very many climates and lands and have listened to many learned men; but no one has ever yet surpassed me in the construction of lines with demonstration; no, not even the Egyptian rope-stretchers with whom I lived five years in all, in a foreign land."*

Eratosthenes was a polymath and coupled with his position it would be reasonable to assume he had knowledge of the manner in which Democritus arrived at his volume theorems and perhaps pass it along to Archimedes. It remains a conjecture but it is general thought that Democritus' method had some similarity to the "Method" Archimedes was using to find mechanical solutions that made geometric proofs easier.

Archimedes gave a hint as to how Democritus might have arrived at his theorems and, supposing Eratosthenes to be his likely source of that knowledge, I wondered if

Eratosthenes might also have learned something of Hippocrates reasoning that gave him a hint for constructing his mesolabe.

Eratosthenes indicates in the *Platonics* that it was Hippocrates who transformed the cube duplication problem into a form that ultimately led to its solution.

> *It used to be sought by geometers how to double the given solid while maintaining its shape. After they had all puzzled for a long time, Hippocrates of Chios was first to come up with the idea that if one could take two mean proportionals in continued proportion between two lines, of which the greater is double the smaller, then the cube will be doubled. Thus he turned one puzzle into another one, no less of a puzzle.*

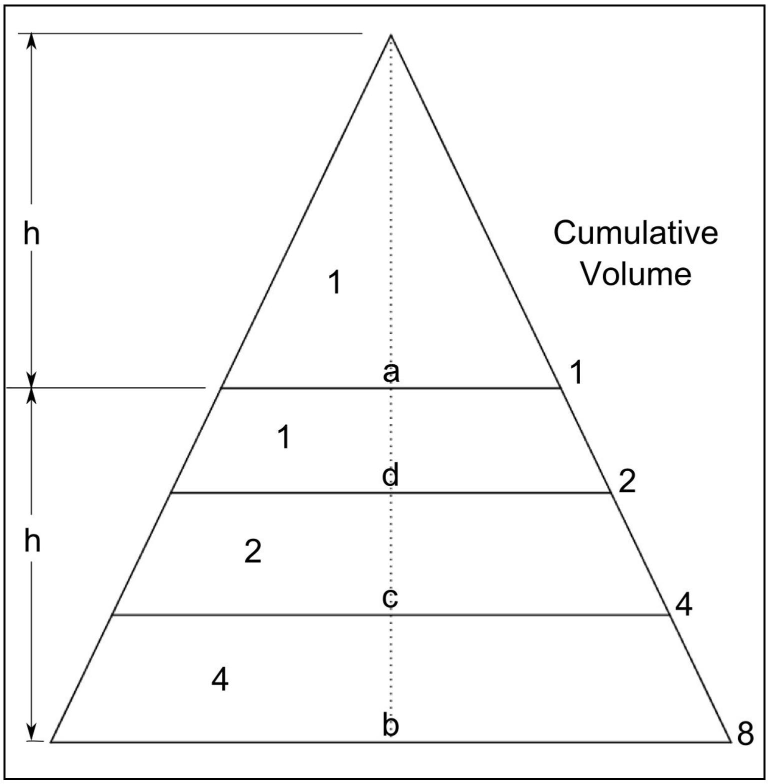
A meaningful translation of mesolabe is "means finder." Descartes many centuries later was influenced by Eratosthenes' device in the construction of his mesolabe compass though it was functionally quite different. Eratosthenes mesolabe was a mechanical device that aligned the sides and placement of three or more similar triangles so that the corresponding sides of the triangles were arranged sequentially by length and in continued proportions so as to create "means between the extremes."

Hippocrates would have been aware that numerically there are 2 means between cubes in a geometric progression of numbers. i.e., 2,4,8,16 where 4 & 8 are the two means between 2 & 16; 1,3,9,27 where 3 & 9 are the two means between 1 & 27. This reminded me of something observed when working on the Egyptian frustum formula that warranted another look. Our knowledge of the Egyptian formula only dates back a hundred years or so when the Moscow papyrus was translated but it would almost certainly have been known to the early Greeks who studied in Egypt.

Referring back to the previous discussion about the Egyptian formula for calculating the volume of a frustum, the volume of a pyramid can decomposed as

$V = V_u + V_f = V_u + 7V_u = (1+1+2+4)V_u$.

This suggest that the pyramid can be sliced into four layers something like is done in the figure shown here.



If s is the seked of our pyramid and h_i is the height of the pyramid whose base is $line_i$ then: $h_i = \frac{line_i}{2s}$.

and the volume of the pyramid whose base is $line_i$ will be: $V_i = \frac{line_i^3}{6s}$ where V_i is the cumulative Volume. From this $\frac{V_i}{V_j} = \frac{line_i^3}{line_j^3}$ or $\frac{line_i}{line_j} = \frac{\sqrt[3]{V_i}}{\sqrt[3]{V_j}}$.

And, finally,

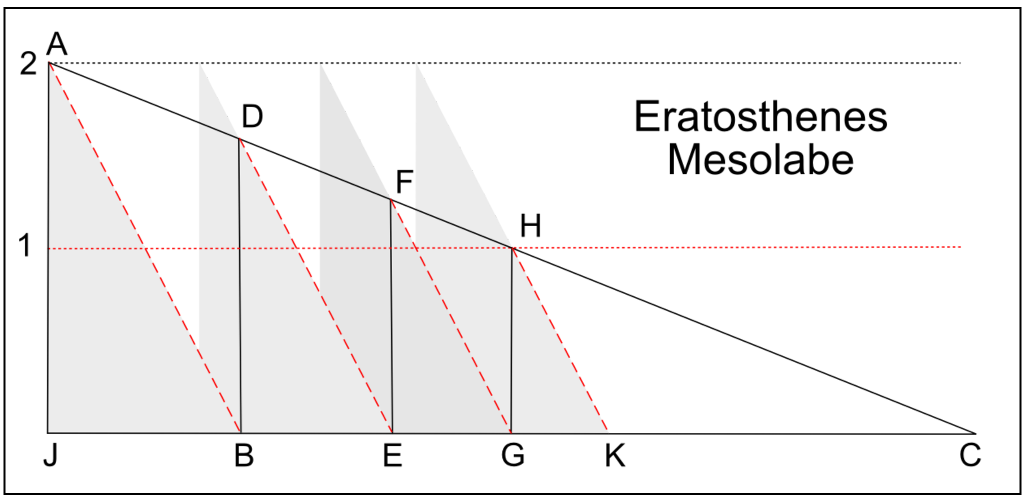
$\frac{d}{a} = \sqrt[3]{2}$; $\frac{c}{d} = \sqrt[3]{2}$; $\frac{b}{c} = \sqrt[3]{2}$.

Nothing to it, except why did Eratosthenes say "*Thus he turned one puzzle into another one, no less of a puzzle*" in reference to Hippocrates?

The obvious answer is likely that if Hippocrates approached the problem in this manner that he didn't know how to position lines c and d to effect the solution. In any pyramid the sides of the base, b, are always twice the length of the sides of the cross section, a, at half height. This makes positioning a easy but he did not know how to position c and d so as to divide the lower part of the pyramid into the three layers that were 1, 2 and 4 times the volume of the upper part. To do this he needed to know how to find two means between 1 and 2.

In Eratosthenes' time, Alexandria was a center of catapult construction and an urgent need existed for reckoning means related to their design. It had been determined that the diameter d of the cord bundle comprising the catapult torsional springs was related to the mass m of the projectile being launched by the formula $d = 1.1\sqrt[3]{100m}$. Ptolemy was his patron thus Eratosthenes likely had some obligation to assist in this military effort. In the library's holdings he may have found something regarding Hippocrates' thinking that led to the development of the mesolabe and solved both the problem that had stymied Hippocrates as well as meeting an immediate need. Catapult design was one application of the mesolabe that Eratosthenes cited though others questioned its suitability for that purpose.

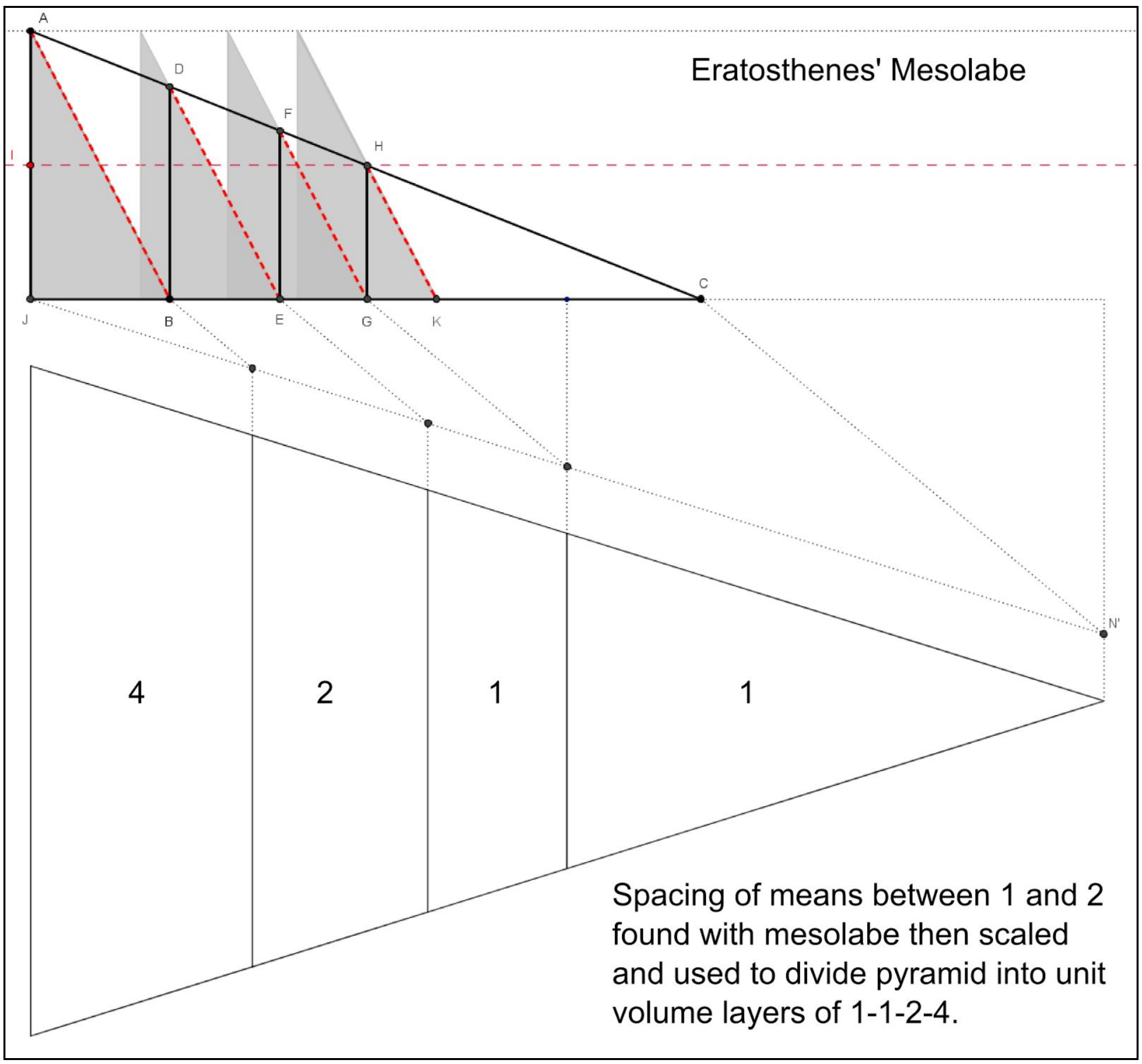
Eratosthenes' mesolabe consisted of equal width, congruent panels which were placed in grooves so that they could slide past each other. The mechanism was constructed so that the points D, F, and H, where line AC contacts the sloping face of the panels, were forced to move vertically so as to always align with the bottom right edge of the



adjacent panel. When line AC was rotated about A it forced the points downward and caused the panels to move to the left which effectively reduced both the width and the height of the active parts of the three movable panels. The two skeins of parallel lines along with the two lines AC and JC create two sets of similar triangles. In the set which are similar to AJC, the hypotenuse of each is also the side opposite the obtuse angle of one of the triangles similar to ABC. From this
JC:BC=AC:DC and AC:DC=BC:EC and when combined is JC:BC=BC:EC;
BC:EC=DC:FC and DC:FC=EC:GC and when combined is BC:EC=EC:GC;
EC:GC=FC:HC and FC:HC=GC:KC and combined is EC:GC=GC:KC.
All three can then be combined as: JC:BC=BC:CE=EC:GC=GC:KC
and for the corresponding sides: AJ:DB=DB:FE=FE:HG
showing that DB and FE are two means between AJ and HG.

Applied to the problem at hand, the fixed panel widths introduce a slight problem. The width of the panels determine the length JC and hence JG. Using an interactive geometry tool such as Geogebra (GGB) the width of the fixed panel is made adjustable and the sliding panels self adjusting to its width. Thus JC can be set to match the height of the pyramid and the panel widths adjusted so that HG aligns with G the midpoint of JC.

Eratosthenes didn't have GGB and whatever the panel width came up with for JC he could have scaled it to fit the pyramid as is shown in this figure.



Whatever its merits as a cube root extractor for "rock throwers," the mesolabe could solve the problem conjectured to have stumped Hippocrates. In the concluding section an even simpler solution is given that Hippocrates Egyptian tutor's should surely been aware of.

As Heath tells it, the Delian problem originated with the story that

> *"an ancient tragic poet had represented Minos as putting up a tomb to Glaucus but being dissatisfied with its being only 100 feet each way; Minos was then represented as saying that it must be made double the size, by increasing each of the dimensions in that ratio. Naturally the poet ' was thought to have made a mistake '. Von Wilamowitz has shown that the verses which Minos is made to say cannot have been from any play by Aeschylus, Sophocles, or Euripides. They are the work of some obscure poet, and the ignorance of mathematics shown by him is the only reason why they became notorious and so survived."*

or, basically it was a math error in a mythical drama.

Heath also relates a story attributed to Eratosthenes

> *"That when the god proclaimed to the Delians by the oracle that, if they would get rid of a plague, they should construct an altar double of the existing one, their craftsmen fell into great perplexity in their efforts to discover how a solid could be made double of a (similar) solid ; they therefore went to ask Plato about it, and he replied that the oracle meant, not that the god wanted an altar of double the size, but that he wished, in setting them the task, to shame the Greeks for their neglect of mathematics and their contempt for geometry."*

Heath finds the story attributed to Eratosthenes creditable but also gives a somewhat different version that is falsely attributed to Eratosthenes. Some think it likely to have been an attempt to use Eratosthenes' stature to enhance Plato's academy's prestige for having solving the problem.

> *"Geometers took up the question and sought to find out how one could double a given solid while keeping the same shape; the problem took the name of "the duplication of the cube" because they started from a cube and sought to double it. For a long time all their efforts were vain; then Hippocrates of Chios discovered for the first time that, if we can devise a way of finding two mean proportionals in continued proportion between two straight lines the greater of which is double of the less, the cube will be doubled; that is, one puzzle was turned by him into another not less difficult. After a time, so goes the story, certain Delians, who were commanded by the oracle to double a certain altar, fell into the same quandary as before."*

With the attention paid to mythical stories regarding the source of the problem and the effort to popularize the plato school geometers who ultimately solved it, it seems only appropriate to include this historical tale - some will say yarn - that suggests an Egyptian catalyst for Hippocrates' idea that the problem of doubling the cube could be solved by finding two means. While technically feasible, that the idea may have originated with something Hippocrates' observed in Egypt is only a conjecture.

A Visit to Ahmes' Paramidia Shop

Knock on door. **Ahmes:** Come in, the door is unlocked. Tourist Enters.

Ahmes: Good day madam. Welcome to Ahmes' Paramidia Shop. May I help you with something?

Tourist: Yes, I was looking to purchase a nice pyramidion for a souvenir.

Ahmes: Would this be a large one for your garden?

Tourist: No, I want to place it on a display stand in my library. I like the one you have on display but I would prefer a larger one? Do you have any that are larger?

Ahmes: If you like the one out front, I can make you one and have it ready tomorrow. We cast them from a copolymer and then buff it to give it a high sheen. We can make it any size you want. The volume of the material determines the price. The one in front is the basic size. Our mold is designed so that we can easily double, triple, or quadruple the size. That's why we are a pyramidia rather than a pyramidion shop.

Tourist: Interesting. We just recently arrived from Chios and while we were there we heard Hippocrates announce that he had reduced the problem of doubling a cube to finding "two means between two extremes." No one seemed to have an idea of how that could be done and we were afraid it might be difficult to double the size of a pyramidion.

Ahmes: That Hippocrates! He comes to town regularly to hang out with the geometers. He seems to like our geometry but refuses have anything to do with our numbers. He seems to think in lines, angles and shapes.

Anyway, he came in here and said he had heard I could make pyramidia of any multiple of the basic size and wanted so see how I did it. I have a model to show customers and as I didn't figure he would become a competitor I showed him my machine model. He found it interesting but he still wanted to see the actual molds we used.

So we put on our hard hats and I took him in to see the molding machine. The model is a cutaway of the real casting machine and I didn't think seeing the actual molds would be that exciting to him. In addition to the basic size, twice, quadruple, and octuple sizes are so common that the craftsmen have marked lines on the molds for those sizes so they don't have to measure the material. They just pour it in until it fills the mold to the desired line.

I didn't know what it meant but when Hippocrates saw those lines he said, "that's it," and started yelling over and over something about "two means between two extremes." After a bit he calmed down, thanked me and left. I hadn't heard that he had returned to Chios.

Tourist: That's even more interesting. Could you show me your molding machine? Or, at least your model?

Ahmes: Certainly. But I don't see what the big deal is. There are six congruent right angled pyramids in a cube. I can double a pyramid so of course I can double a cube.