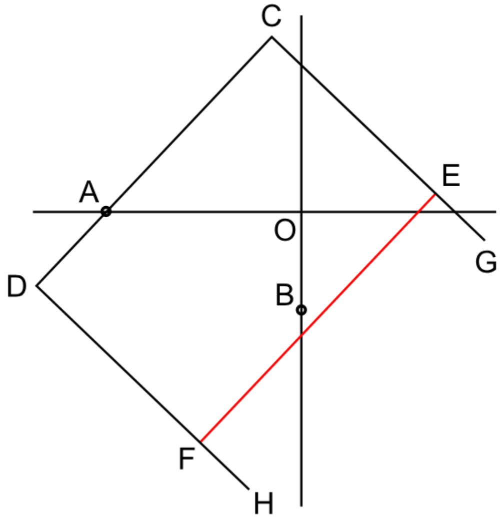
From Plato’s Frame to Menaechmus’ Conics

Eutocius, writing in the 5th century AD, described several solutions to the problem of find two means in his commentary on Archimedes’s *Sphere and Cylinder* though Archimedes himself did not propose a solution. Interestingly Eutocius mentioned Eudoxus but claimed Eudoxus’ solution, at least in the form available to him, was invalid. He also detailed a solution attributed to Plato for which there is no other known historical record and the attribution is generally regarded as lacking credibility.

Plato’s solution may be incorrectly attributed, but the solution itself is valid and may well have contributed to the solutions by Eudoxus and/or Menaechmus. Eudoxus’ solution is unknown but a possible solution using a curve now known as Eudoxus’ Kampyle was presented earlier. While the historical references to Eudoxus’ proficiency as an instrument maker may add credence to the previous compass based solution, a plausible solution using curved lines and derived from the Plato device is presented as a simpler solution in this section. Menaechmus’ solution seems almost certain to have either been derived from the device or conversely it may have been the basis for the device.

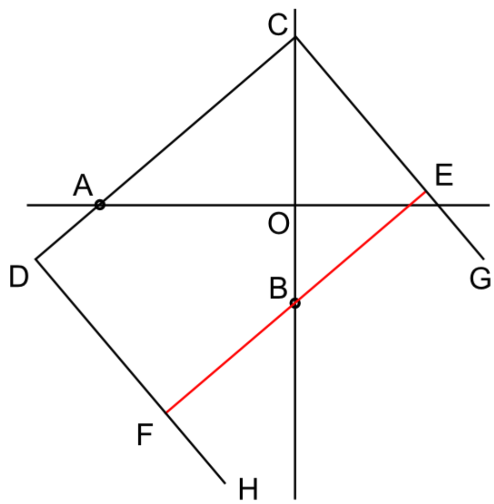
Plato’s Mean Finding Frame



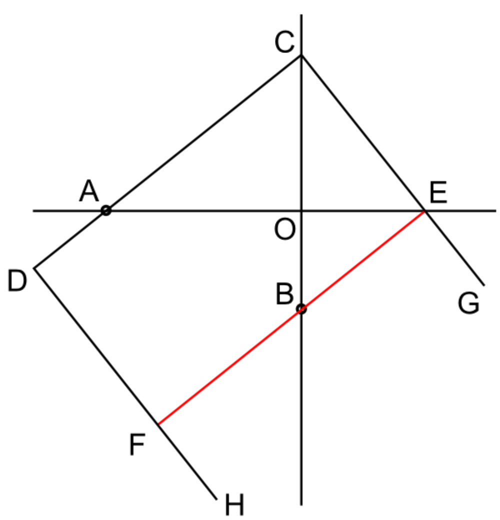
The device for finding two means between two extremes that Eutocius attributes to Plato is a U-shaped frame with an adjustable sliding bar across its mouth.

It is set up for use by drawing two perpendicular lines and marking off a length OA from their intersection on one line and OB on the other where OA and OB are the two lengths between which two means are sought. Aligning the frame to find the means is easier if pins are place at A and B.

The sliding bar FE is then adjusted so that the frame HDCG can be placed over the two pins with side DC against pin A as shown.



DC is then held against pin A as it is rotated it until C lies on BO extended. The bar is then adjusted so that it touches pin B.



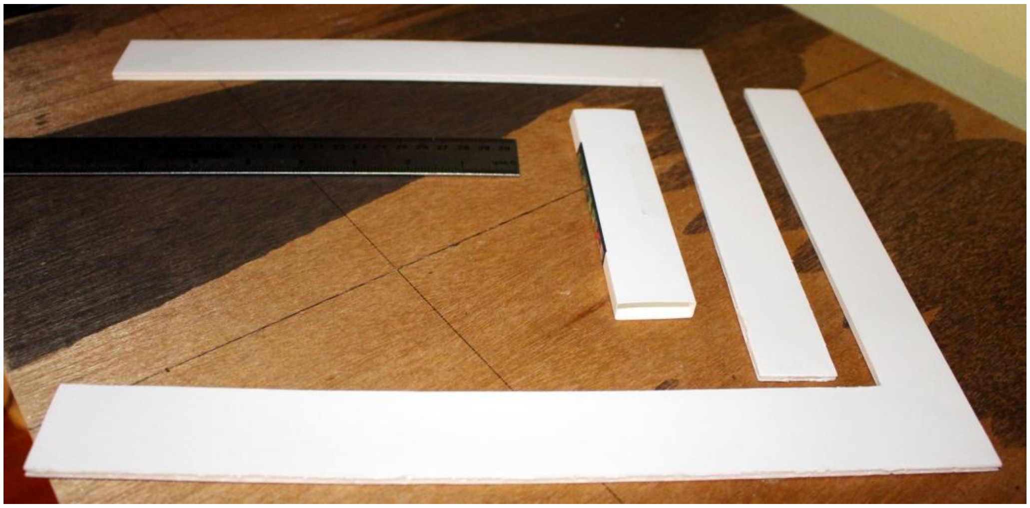
Holding the frame against pin A and keeping C on BO extended, adjust the bar as the frame is slid downward until E lies on AO extended and the bar is against pin B as shown.

CO and OE are the required two means between AO and BO.

The usual proof is based on the theorem that the altitude of a right triangle divides the hypotenuse into two sections and it is the mean between those sections. From this,

$\frac{AO}{CO} = \frac{CO}{OE}$ and $\frac{CO}{OE} = \frac{OE}{OB}$; or, $\frac{AO}{CO} = \frac{CO}{OE} = \frac{OE}{OB}$

Arm DH of the frame serves only to keep the movable bar FE parallel to DC. A simplified version can be made from a pair of foam board carpenter squares and a card stock sleeve. The example shown here is used to finding the two means

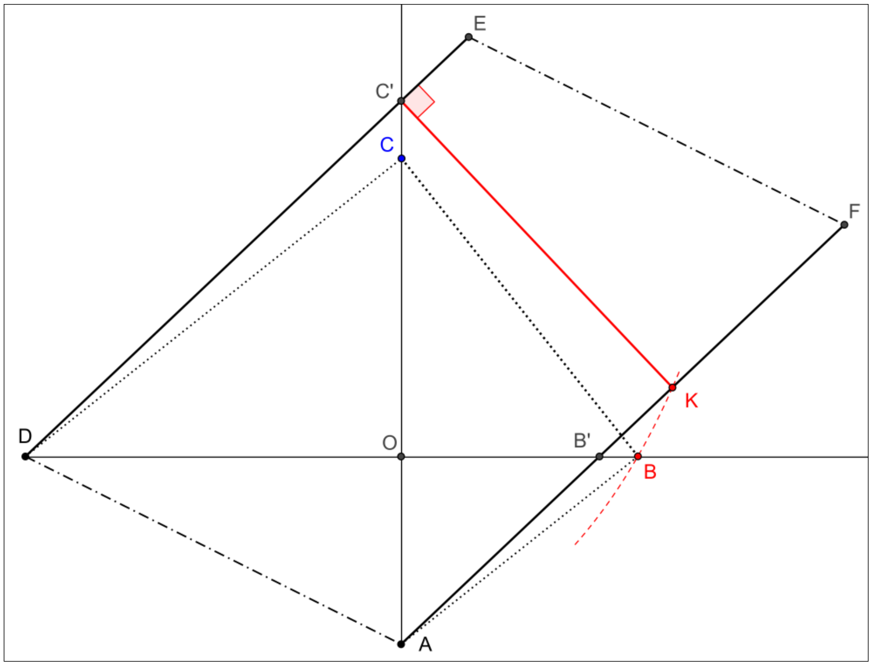


between 200 and 100 mm.

As can be seen in the next photo, the sleeve was cut into two parts and a part of the sleeve attached to each short arm near the right angle to keep them parallel as

they slide in the sleeves. Then, as with the frame, one leg is held against the pin at D and corner at the opposite end is aligned with point C located on AO extended and the other arm is slid up to touch pin A. The whole device is then slid downward while rotating it to keeping point C on the line and adjusting the other arm to keep it against pin A until the corner on that arm, point B, aligns with the line DO extended as shown. OB and OC are then the required means between AO and DO.

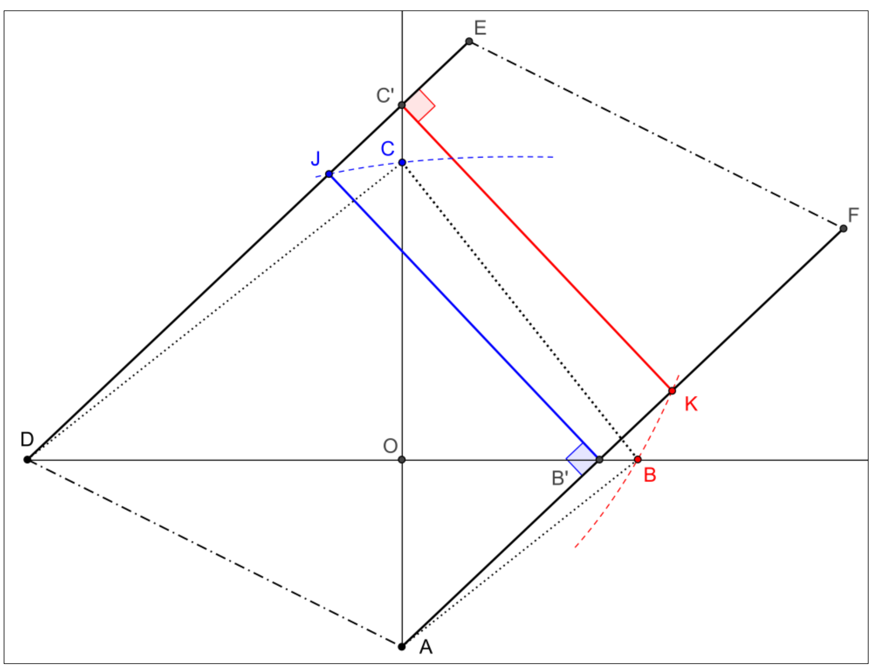
Consider now if a parallelogram is constructed with sides DE and AF with points B’ and C’ attached so that they are constrained to move along AO and DO and the sides of the parallelogram to which they are attached as shown in this figure.



Vertices A and D are fixed and make AD stationary but rotating DE about D rotates the opposite side of the parallelogram AF about A so that the sides remain parallel as they rotate.

C’K is perpendicular to and attached to DE at C’ and intersects AF at K. Since DE and AF are parallel C’K is also perpendicular to AF.

Rotating the parallelogram so that the sides DE and AF rotate clockwise moves B’ away from O along DO extended while C’ moves downward toward O and line C’K rotates toward B. Point K moves along FA but because the lines are rotating Point K actually follows the red curved path with K and B’ intersecting at B. When this occurs, C’ will be located at C and the lines joining DCBA will be aligned as with the modified frame.



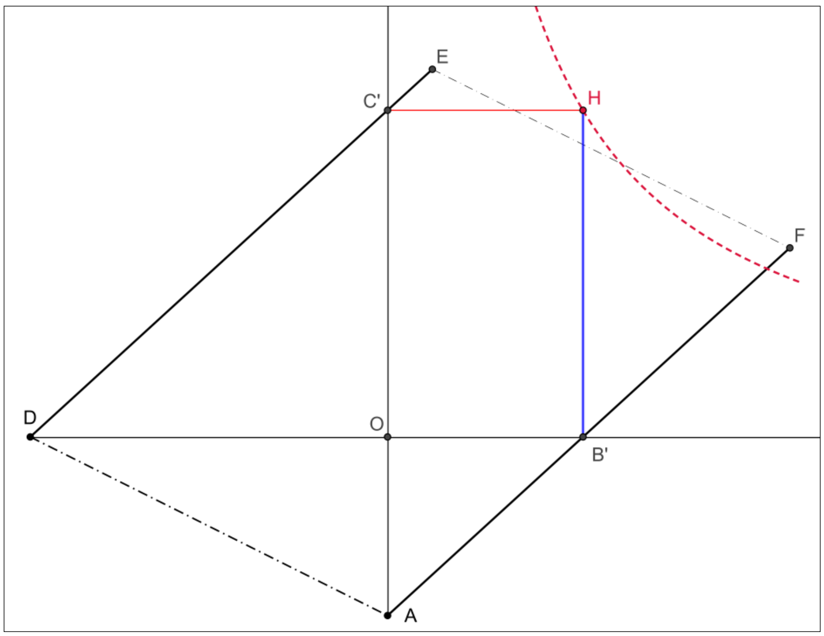
Going a step further, B’H is added perpendicular to AF at B’ and intersecting DE at J. Rotating F produces the blue curved path as J moves and crosses AO at C where J and C’ coincide.

Thus the intersection of the curves KB’ and JC’ with AO and DO extended locate the means.

For those skeptical of the “Kampyle” being Eudoxus’ solution of the cube problem, this offers an alternate possibility that some may find more consistent with Eratosthenes reference to “...that shape curved in the lines by the God fearing Eudoxus.”

While the two curves above found the two required means the curves are unique to the device and cannot be created by any other means. With a slight modification, however, the device can construct two curves which both solve the problem and can be constructed by other means.

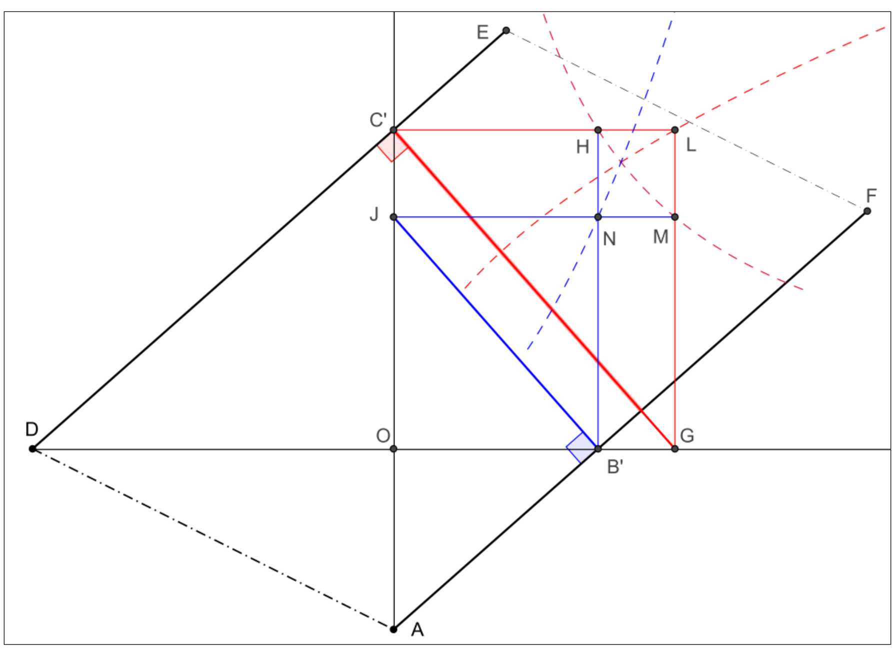
We temporarily remove the two arms that were perpendicular to AF and DE and replace them with ones that are perpendicular to AO and DO extended. C'H and B'H in this drawing. Tracing the path of point H as DE and AF are rotated results in the crimson curve passing through H.



Since DE and AF are parallel they divide AC' and DB' proportionally. That is $\frac{AO}{OB'} = \frac{OC'}{DO}$ and, by re-arranging,

$$OB' \cdot OC' = AO \cdot DO.$$

That is, the area of the rectangle created by the lines cutoff on AO and DO extended by lines dropped perpendicular to them from a point H on the curve is equal the area of the rectangle with sides DO and AO. These are known and constant and when H is correctly positioned OB' and OC' will be the required means. Unfortunately there is insufficient information to position H so back to the drawing board.



We draw the red line C'G and draw a line parallel to OC' through G intersecting C'H extended at L. The path of L as DE and AF are rotated is traced producing the red curve.

At the point where H and L overlap C'G and JB' are collinear and the ordinates of H (and L) are the two means.

Notice that M produces the same curve as H and N produces a curve that intersects the curved generated by H and L at the point that they intersect. Thus any two of the three curves can be used to find the required means.

The wisdom of the internet is that Menaechmus did not have a mechanical device for drawing these curves. I think it likely that he did have such a “compass” or a mesolabe as both Eratosthenes and Descartes two millennia later called them but, as with many other things, there is no definitive answer.

Now OC’:OG::JO:OB’ from which OJ*OG=OB’*OC’=AO*DO. The area of the rectangle created by the lines cutoff on AO and DO extended by lines dropped perpendicular to them from a point M on the curve is equal the area of the rectangle with sides DO and AO just as was proven for point H.

On the other hand, the altitude from the side opposite the right angle of a right triangle divides the side into two segments and is the mean between the two sides. That is OD:OC’::OC’:OG and AO:OB’::OB’:OJ. But OJ=NB’ and OC’=GL thus L is the plot of the means between DO and OG while N is the plot of the means between AO and OJ.

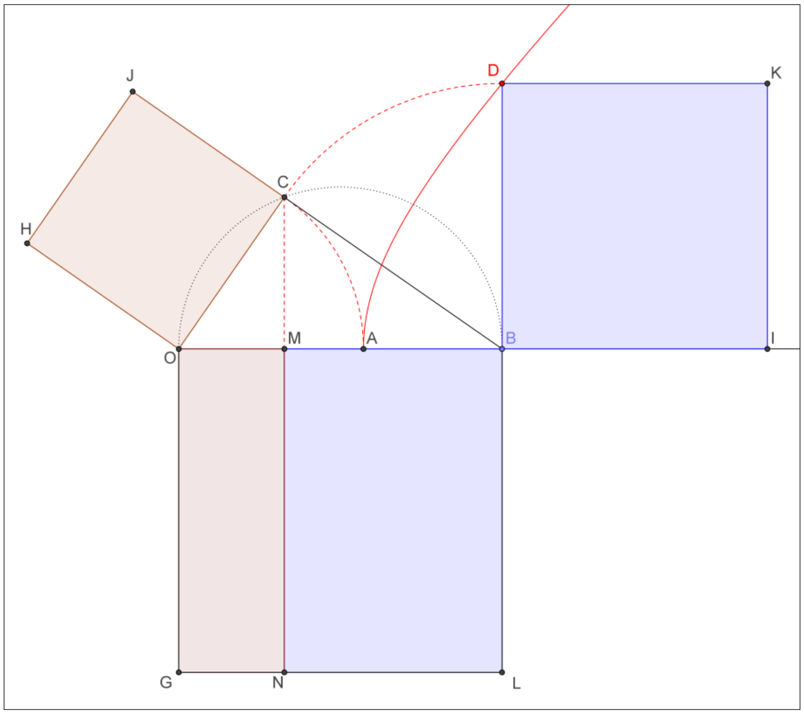
Since the properties of right triangles had studied extensively by the Pythagoreans it seems reasonable to assume these curves were recognized by Menaechmus. How they became associated with right-angled and obtuse-angled cones as they were identified in Euclid Elements a few years latter is less obvious but mainly because of the geometrical properties of right triangles and the method of application of areas is unfamiliar to modern readers.

For example, the numerical version of the Pythagorean theorem, “the square of the hypotenuse is equal the sum of the squares of the sides forming the right angle” is well known, but the geometrical version in Euclid’s Elements VI.31 is not.

> *“In right-angled triangles the figure on the side opposite the right angle equals the sum of the similar and similarly described figures on the sides containing the right angle.”*

Restated for our purposes this becomes “the figure on one of the sides containing the right angle equals the difference between the figure on the side opposite the right angle and the figure on the other side that contains the right angle.”

In this figure, angle OCB is a right angle as indicated by point OC lying on



semicircle OB.

OC has a fixed value, f. Point B was initially positioned at point A where OC lay along OA and OA=OC= f and BC=0.

As B is moved to the right the length of OB increases as does the length of BC. This in turn increases the area of the squares on both OB and BC. Because Angle OCB is fixed, the increase in BC causes OC to rotate about O but the square on OC does not change.

The square on OB is comprised of the square on OC plus that on BC. Thus, the square on BC is the difference between the square on OB and the square on OC.

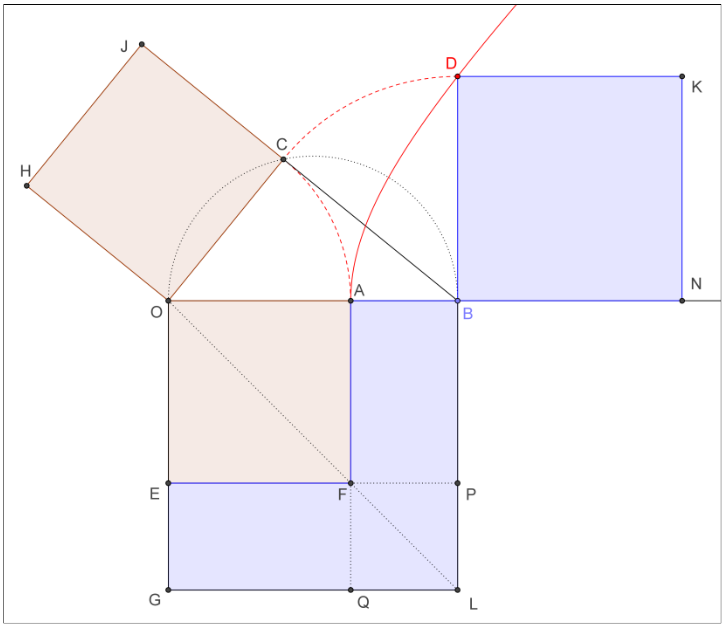
The square on OB is shown divided into areas that are equal the squares on OC and BC. DB has been constructed perpendicular to OB and equal BC. Hence the area of the square on BD equals the area of the square on OB that corresponds to the area of BC.

The trace of point D then generates a curve AD whose ordinate D at any point on the curve has a length such that the square on the ordinate equals the square on OB less the square on OC=OA=f.

This leads to an obvious observation. The blue area of the sq OB is the rect MB*OB=BC*BC when CM is the altitude. That is BC is the mean between MB and OB. While this is true for any right triangle, remember that in this application OC is fixed.

Many of the properties of right triangles were known to the Pythagoreans and at some point they also developed what is now referred to as the application of areas to a line. Like many other facets of ancient geometry, the impetus for its development is unknown and may have resulted from the general advancement of the art. It was used for determining and verifying the properties of curves but the extent to which it may have been used for the drawing of curves is unknown.

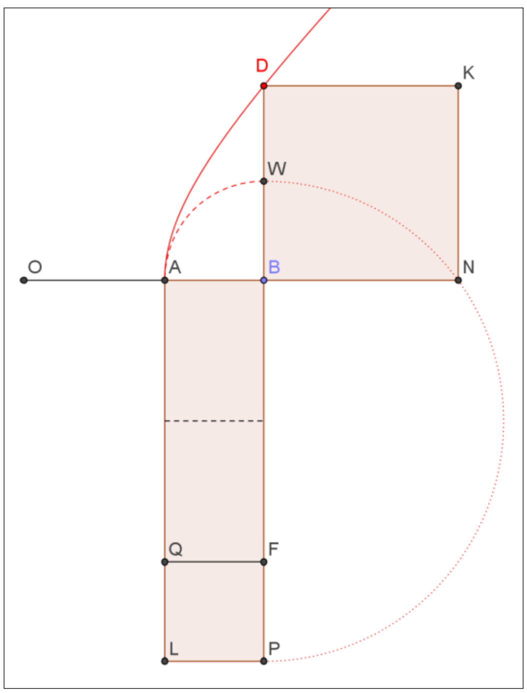
Apollonius would use it later to show that the sections cut from an arbitrary cone were the same as the curves from the earlier era. We show how it is related to the right triangle method and use it to find an obtuse cone containing a given section.



The first step is to repartition the area on the hypotenuse. The slice along OG that was the area of the square on OC is replaced with the square OAFE having the same area.

AF is extended to intersect GL at Q and EF to intersect BL at P. The diagonal of OF lies on the diagonal of the square OL and AFEGLB is a gnomon of area = the area of the square on the ordinate BD=BC.

To eliminate the right triangle stuff requires an alternate method of constructing a rectangle on AB and AQ extended with an area equal that of the gnomon for all AB. You can probably see how this needs to be done but to make it look authentic we quote Euclid II.4

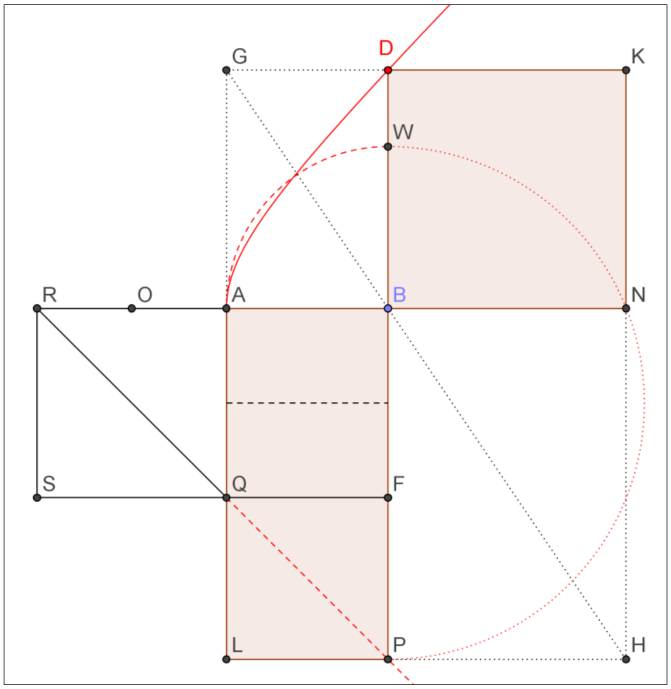


"If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts.

The square on OB can thus be viewed as consisting of the square on OA plus the gnomon confirming our observation above and the right triangle components can be replaced with two rectangles having one side of fixed length OA and the other side of length AB plus a square with sides of length OA.

This reduces the model to that shown at the left where rect AP consists of a square AB*AB and two rectangles OA*AB which now needs to be converted into the sq BK with the same area.

This is done by drawing arc ABW to extend PB by the length AB to point W. A semicircle with PW as diameter is then used to find BN the mean between PB and BW. BD is then made equal to BN. The square on BD now equals the area of the rect AP which equals the area of the square on OB less the square on OA as it did when using the sides of the right triangles to calculate BD.



The trace of D is now congruent with that generated by using the right triangle procedure which achieves the original goal. However, by adding a few lines more information about the curve becomes available.

To this end, diagonal PQ is extended to intersect AO extended at point R. RS and RQ are added to complete the sq RQ which has sides of length 2OA.

RA extended is the diameter of the curve, BD is the ordinate and AB is the part of the diameter cut off by the ordinate.

Using this nomenclature we can describe our curve with the assertion that “the square on the ordinate is equal to the rect AB,BP applied to the line AQ and exceeding it by a square.” The “applied to line” AQ is called the parameter.

This seems strange a modern reader familiar with algebra but another two millennia will pass before algebra shows up on the time line. For the geometer of that time the ratio AQ:RA is the ratio of the square on the ordinate to the rectangle formed by the diameter RB and the part AB that is cut off by the ordinate.

That the diagonal of the square QP lies along the extended diagonal of RQ shows that the area of the square on the ordinate is increasing but that the area has a constant ratio to the area of the rectangle on AB. i.e., BD*BD:AB*BP=AQ:RA.

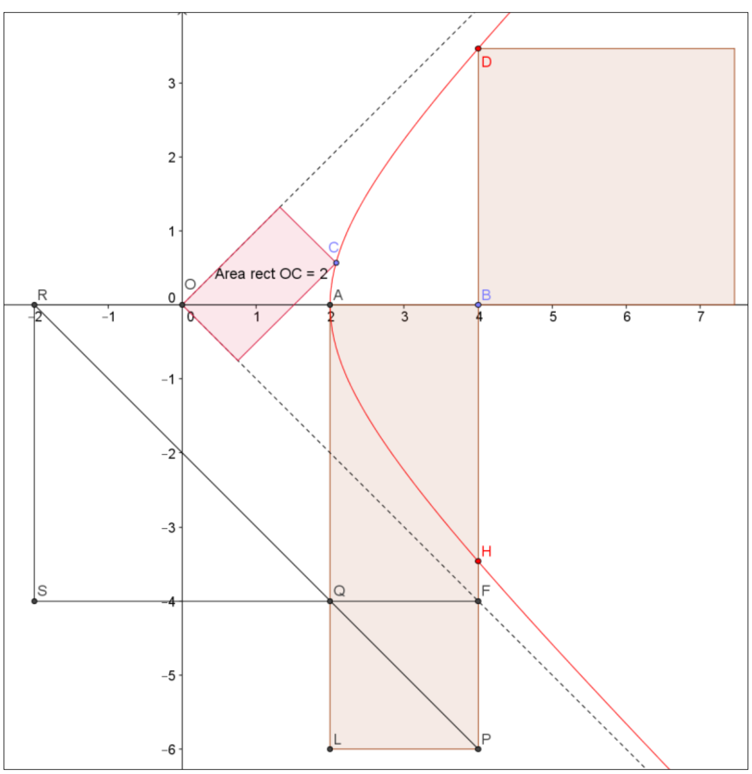
In “*The Works of Archimedes*” Heath asserts that “It may be taken as certain that the following properties … were proved in the conics of Euclid.” This item is included in his list.

> *In a hyperbola, if P be any point on the curve and PK, PL be each drawn parallel to one asymptote and meeting the other PK*PL = (constant.)*
> *This property, in the particular case of the rectangular hyperbola was known to Menaechmus.*

This property may have been the basis for connecting the curve created by the frame device with the curve generated using a right triangle, or, the connection could have been established by aligning the axes of the curves and thereby discovered this property for the AOA curves.

There are contradictory accounts but Plato seems to have not been pleased with the solutions Archytas, Eudoxus and Menaechmus devised to solve the Delian problem. Eratosthenes’ statement to “…not cut the cone in the triads of Menaechmus” implies Menaechmus may have resorted to actually slicing a cone to produce the curves he needed. The AOA method provides all the information that is needed to find a right cone containing a given curve using a cutting plane that was perpendicular to a cone generator.

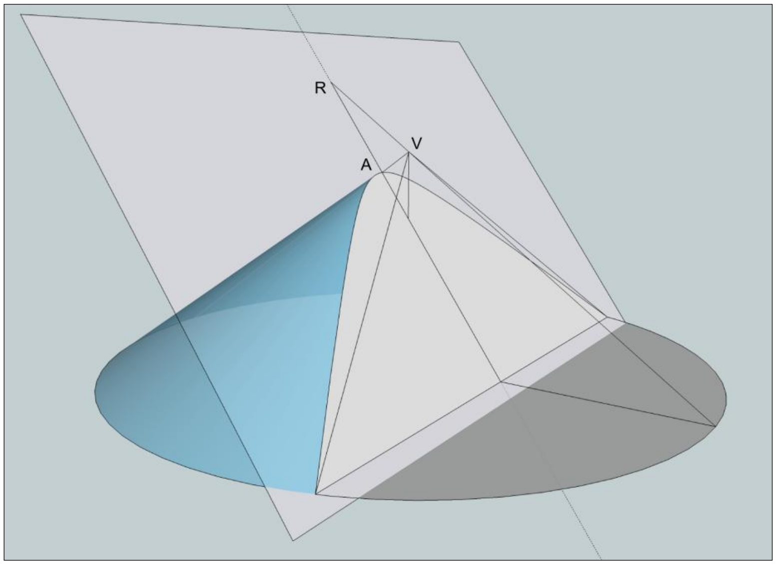
Some dimensions are needed to construct a cone and cut it so that the section is the desired curve. Thus in this figure the we show the measurements on an AOA



model that has been adjusted to match our plato model for finding the two means between 1 and 2. The goal here is to find a cone and section so that the curve from the cone matches this curve.

Our solution will be similar to the section shown in the cutaway view on the following page. The cutting plane is perpendicular to a cone generator and is also perpendicular to the plane that both that generator and the cone axis lie on called the axial triangle plane.

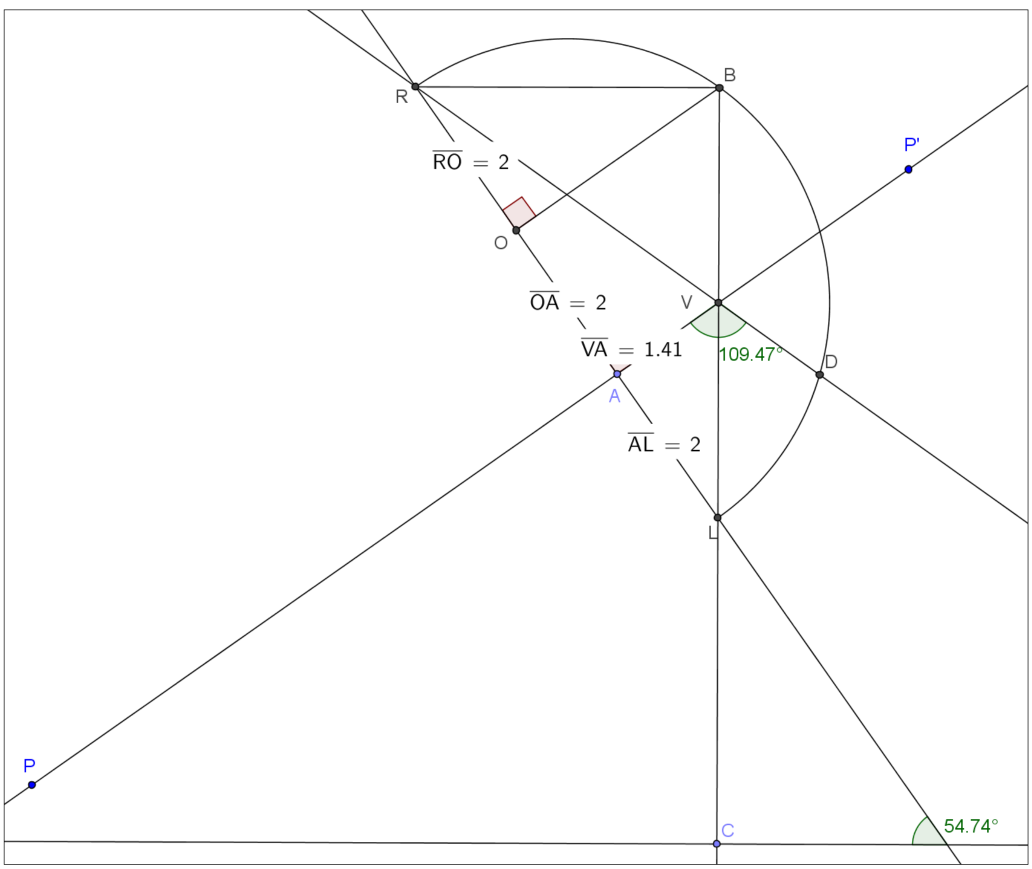
The diameter of the curve lies along the intersection of the cutting plane and the axial triangle and thus it too is perpendicular to a cone generator.



The curve vertex, A, is the intersection of the curve axis and the cone generator line. Point R is at the intersection of the curve axis and a cone generator external to the cone. (It is actually a generator of the upper nape of the cone which is first documented in Apollonius' works.) Point O lies mid-way between points A and R.

Finding the solution involves finding the cone vertical angle, constructing a cone with that angle, finding the cutting plane location and finally cutting the section from the cone. The properties of the curve as taken from the AOA model are used to calculate the properties of the cone and the angle and location of the cutting plane.

Cone and cutting plane design.



Let PP' be a line which will become side of the axial triangle of the cone.

Draw PP' diagonally so that P' is higher than P.

Using data from the AOA model locate a point A on PP' so that AP' ~ 2*OA.

Construct a line through A perpendicular to PP'.

Locate points R and O on the perpendicular line above PP'.

Locate point L on the opposite leg of the perpendicular so that AL=OA.

Construct a semicircle with RL as diameter on the P' side of RL.

Construct a line perpendicular to RL intersecting the semicircle at B.

Draw a line connecting RB and another connecting BL. Angle RBL is right.

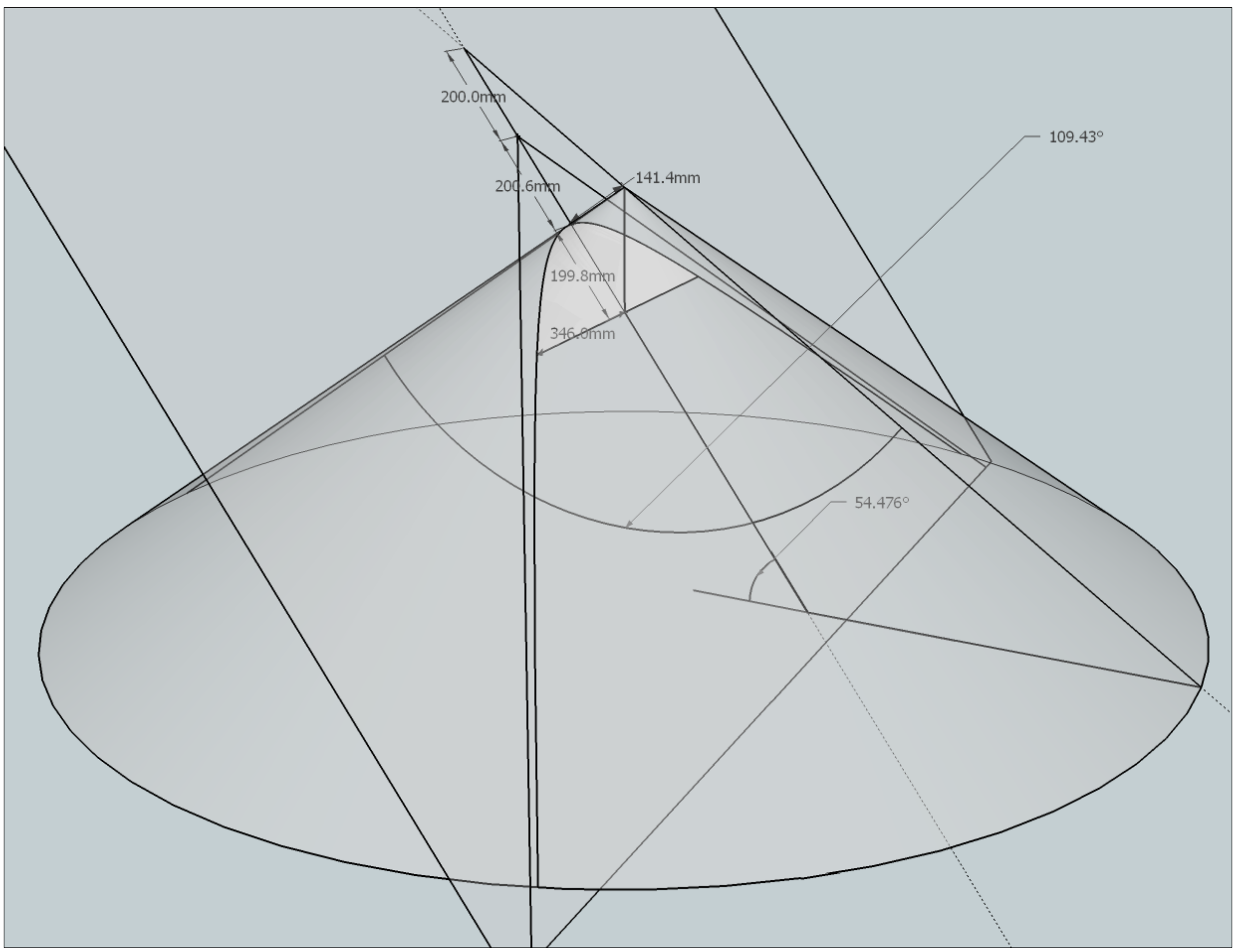
BL extended is the axis of the cone.

Locate point V at the intersection of PP' and the line BL.

Draw RV. RV extended is the other side of the axial triangle.

This cone, cutting plane and section as constructed using SketchUp is shown in the next figure. Measurements and the asymptotes are also shown.

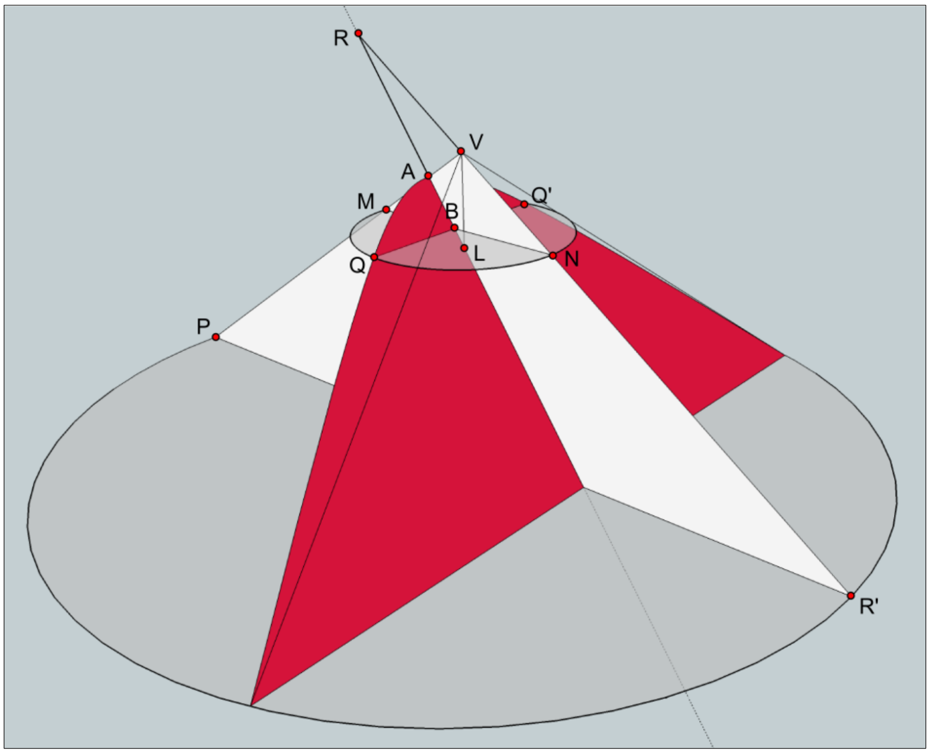
Section cut from Obtuse Cone



Comparing the dimensions from the cone with the ones we calculated there is good agreement between the cone and section with the AOA model. While this is a good indication that the section was successfully created it does not prove that the curve is a hyperbola.

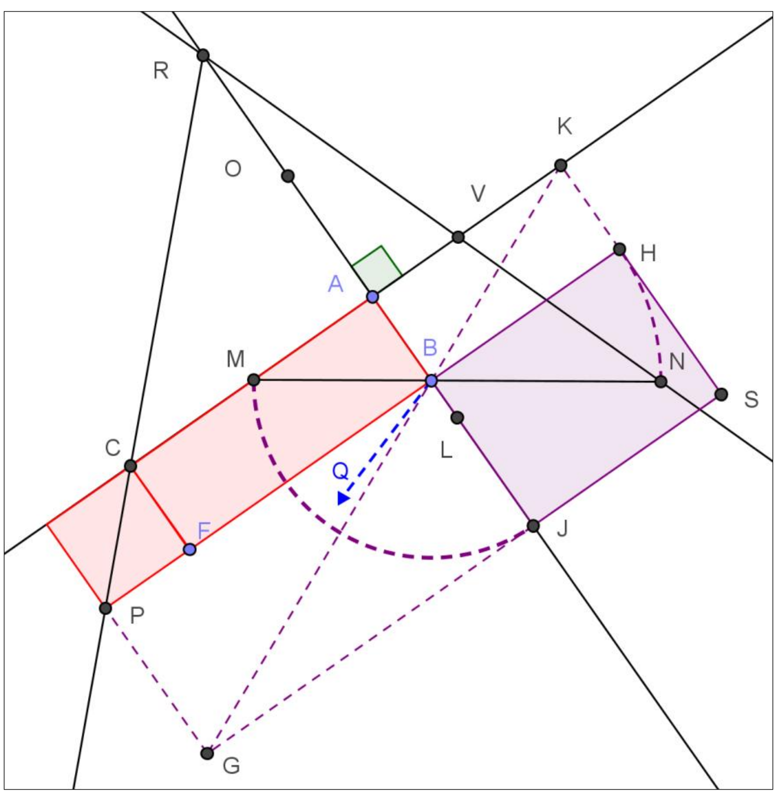
A geometric proof is needed. i.e., it must be shown that for any arbitrary point on the curve that when the square on the ordinate is applied to the parameter as a rectangle having a width equal the distance from the vertex cutoff on the diameter by the intersection of the ordinate with the diameter it exceeds it by a square. To do this we essentially reverse engineer the Application of Area method use for designing the cone.

This is especially easy to do when the section plane is perpendicular to a cone generator as is the case here.



Choose Q to be the arbitrary point on the curve and cut the cone at that point with a plane parallel to the base. This produces a circle with diameter MN centered on the axis of the cone VL. This makes the ordinates of the curve at B, coincident with the chord QQ’ of the circle and thus QB2=QB*QB’=MB*BN

On the axial triangle we construct rect BH,BJ with BH=BN and BJ=BM and then apply the area of rect BH,BJ to AC with a width of AB.



SH is then extended to intersect CV extended at K and KB is drawn and extended to intersect SJ extended at G.

GI is drawn parallel to the diameter RB intersecting HB extended at P. CF is drawn parallel to AB and RC is extended through P.

KG is the diagonal of rectangle KG therefore triangle KSG = triangle KIG. Similarly triangle KHB =

triangle KAB and triangle BJG = triangle BPG. Subtracting the area of these equal triangles from the area of the bigger triangles of which they are part leaves rect.BH,BJ and rect.AB,BP which are thus equal.

QB^2=BH*BJ=AB*BP and AB*BP has been applied to AC exceeding it by the square CP. The section cut from the cone has thus been shown to be the same as those created using the AOA method, the right triangle method and the parallel lines version of Plato's frame and is the curve called a section of an obtuse-angled cone.

A similar process can be used to show that the other two curves drawn by Menaechmus' mesolabe have the property that the area of the square on the ordinate is exactly equal to the area applied to the line AC with a width AB. Both of these curves are thus "sections of a right-angled cone.

Archimedes may have been the first to define a cone as having a circular base. The cone itself could be elliptical but if the base was not circular he called it a segment of a cone. He devised methods of constructing bases that converted segments of a cone into cones. Even so he continued to refer to the curve produced by cutting a cone with a plane as if they had been cut from an xxx-angled cone.

Apollonius too required that a cone have a circular base but he also required that the cutting plane be perpendicular to a diameter of the base or to an extension of a diameter. Then using the properties of the cone and the angle the cutting plane made with the base the sections could be characterized by a diameter, a parameter and the angle between the principal diameter of the section and the ordinates.

This was sufficient to enable the section to analyzed using the AOA method to determine if when the area of the square on the ordinate was applied to the parameter it "fell short", "equalled" or "exceeded" the parameter and thereby showed that the curves were the identical to those created by various methods.

Apollonius said he would call those that fell short, ellipse, those that equaled, parabola and those that exceeded hyperbola. Whether these are simply the translation from the Greek of those terms or whether Apollonius is saying that he has shown that they are the same curves so he will call them by the same names someone had previously given them cannot be conclusively determined.

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